Structural Dynamic Response under Uncertainty - An Interval Finite Element Approach

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Outline

- Introduction
- DFT-Based Dynamic Solver
  - Deterministic Solver
  - Interval Solver
- Examples
- Conclusions

Introduction- Uncertainty

- Uncertainty is unavoidable in engineering system
  - Structural mechanics entails uncertainties in material, geometry and load parameters (aleatory-epistemic)
- Probabilistic approach is the traditional approach
  - Requires sufficient information to validate the probabilistic model
  - What if data is insufficient to justify a distribution?

Introduction- Interval Approach

- Only range of information (tolerance) is available

$$t = t_0 \pm \delta$$

- Represents an uncertain quantity by giving a range of possible values

$$t = [t_0 - \delta,\ t_0 + \delta]$$

- How to define bounds on the possible ranges of uncertainty?
    - experimental data, measurements, statistical analysis, expert knowledge

Introduction- Why Interval?

- Simple and elegant
- Conforms to conventional tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set, imprecise probability)

- **Provides guaranteed enclosures**

Introduction- Interval Arithmetic

- Interval number represents a range of possible values within a closed set

$$\boldsymbol{x} \equiv [\underline{x}, \overline{x}] := \{x \in R \mid \underline{x} \le x \le \overline{x}\}$$

Introduction- Interval Arithmetic

Let x, y and z be interval numbers

1. Commutative Law

$$x + y = y + x$$

$$xy = yx$$

2. Associative Law

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

3. ***Distributive Law does not always hold,*** *but*

$$x(y + z) \subseteq xy + xz$$

Interval in Solid and Structural Mechanics

- Static Linear Problems – Trusses and Frames
- Static Linear Problems – 2 D Continuum Problems
- Static Linear Problems – Plates
- Dynamic Linear Problems – Time and Frequency Domains
- Material and Geometric Nonlinear Problems
- Inverse Problem
- Linear systems with P-Box Coefficients

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DFT-Based Dynamic Solver

- Governing equation of the system

$$\mathbf{K}\mathbf{u} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}$$

- *Bold non-italic font for interval variables*
- Non-trivial initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_0,$$

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Deterministic Dynamic Solver

- Discrete Fourier Transform (DFT)
  - From time domain to frequency domain
  - Non-recursive, obtain solutions at all time steps simultaneously

$$\mathscr{F}_t(u)_j = U_j = \sum_{k=0}^{N-1} u_k \exp\left(-i\frac{2\pi}{N}jk\right)$$

Deterministic Dynamic Solver

- Inverse Discrete Fourier Transform (IDFT)
  - From frequency domain to time domain

$$\mathscr{F}_t^{-1}(U)_j = \frac{1}{N}\sum_{k=0}^{N-1} U_k \exp\left(i\frac{2\pi}{N}jk\right)$$

- Governing equation in the frequency domain:

$$\left(-\omega_j^2 M + i\omega_j C + K\right)\mathscr{F}_t(u)_j = \mathscr{F}_t(f)_j$$

Deterministic Dynamic Solver

- Inverse Discrete Fourier Transform (IDFT)

$$u_k = \frac{1}{N}\sum_{j=0}^{N-1} G_j \mathscr{F}_t(f)_j \exp\left(-i\frac{2\pi}{N}jk\right)$$

- Symmetric conjugate

$$G_j = \begin{cases} \left(-\omega_j^2 M + i\omega_j C + K\right)^{-1}, & 0 \le j \le N/2 \\ \text{conjugate of } G_{N-j}, & N/2 \le j \le N \end{cases}$$

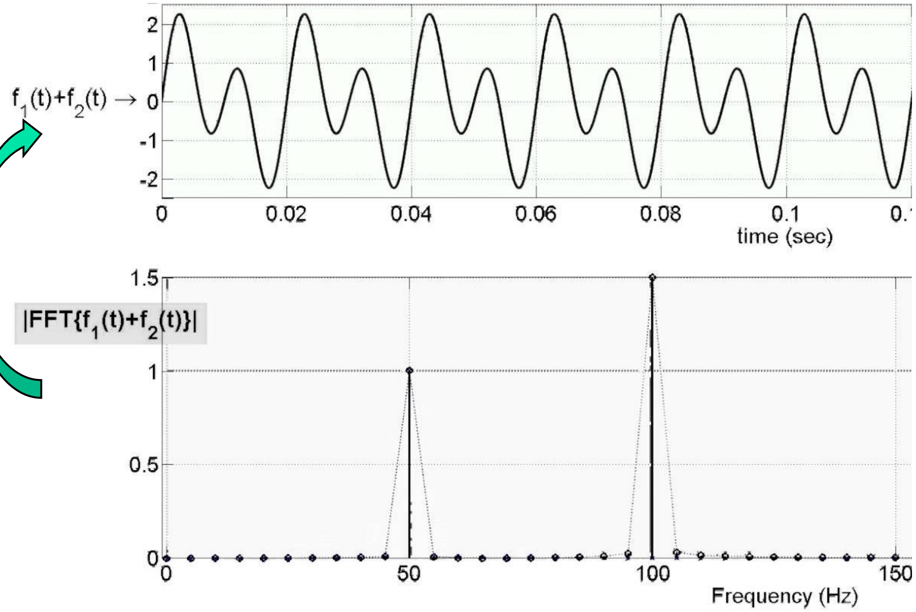
Deterministic Dynamic Solver

- Non-trivial initial conditions:
  - Displacement: constant load
  - Velocity: impact load

Deterministic Dynamic Solver

- DFT and IDFT
  - Time domain ⟷ Frequency domain

$$\mathcal{F}_t^{-1}(U)_j = \frac{1}{N}\sum_{k=0}^{N-1} U_k e^{\left(i\frac{2\pi}{N}jk\right)}$$



$$\mathcal{F}_t(u)_j = U_j = \sum_{k=0}^{N-1} u_k e^{\left(-i\frac{2\pi}{N}jk\right)}$$

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Interval Dynamic Solver

- The interval governing equation

$$\mathbf{K}\mathbf{u} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}$$

- Initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_0,$$

- Assume Rayleigh damping

$$\mathbf{C} = \alpha_d \mathbf{M} + \beta_d \mathbf{K}$$

Interval Matrix Decomposition

- Interval matrix decomposition: reduce multiple occurrence of the same variable
  - Stiffness

$$\mathbf{K} = A\,\mathrm{diag}\left(\Lambda\boldsymbol{\alpha}\right)A^T$$

  - Mass

$$\mathbf{M} = A_m\,\mathrm{diag}\left(\Lambda_m\boldsymbol{\alpha}_m\right)A_m^T$$

  - Equivalent load

$$\mathbf{f} = F\boldsymbol{\delta}_t = \left(F\boldsymbol{\delta}\right)\mathbf{d}_t$$

Interval Matrix Decomposition

- For non-trivial initial conditions

$$\mathbf{f} = \mathbf{K}\mathbf{u}_0 d_{u0} + \mathbf{M}\mathbf{v}_0 d_{v0}$$

- Apply the decomposition of stiffness matrix K and mass matrix M, the equivalent load associated with initial condition is

$$\begin{aligned}\mathbf{f} = \mathbf{A}\left(\Lambda \boldsymbol{\alpha} \circ A^T \mathbf{u}_0\right) d_{u0} \\ + A_m\left(\Lambda_m \boldsymbol{\alpha}_m \circ A_m^T \mathbf{v}_0\right) d_{v0}\end{aligned}$$

Interval Governing Equation

- Apply DFT to transform the governing equation into the frequency domain

$$\left(-\omega_j^2\mathbf{M}+i\omega_j\mathbf{C}+\mathbf{K}\right)\mathscr{F}_t(\mathbf{u})_j=\mathscr{F}_t(\mathbf{f})_j$$

- Procedure:
  - 1) Transform the governing equation to a fixed-point form
  - 2) Use iterative enclosure method to solve it

Interval Governing Equation

- To keep the decomposition, introduce constraints in the form of Cu = 0.

$$\begin{Bmatrix} \mathbf{K}_{\mathrm{eff},j} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathscr{F}_t(\mathbf{u})_j \\ \mathscr{F}_t(\boldsymbol{\lambda})_j \end{Bmatrix} = \begin{Bmatrix} \mathscr{F}_t(\mathbf{f})_j \\ 0 \end{Bmatrix}$$

- The effective stiffness matrix

$$\mathbf{K}_{\mathrm{eff},j} = \begin{cases} -\omega_j^2\mathbf{M} + i\omega_j\mathbf{C} + \mathbf{K}, & 0 \le j \le N/2 \\ \text{conjugate of } \mathbf{K}_{\mathrm{eff},N-j}, & N/2 \le j \le N \end{cases}$$

Interval Governing Equation

- Decomposition of the effective stiffness matrix

$$\mathbf{K}_{\mathrm{eff},j} = A_{\mathrm{eff}}\,\mathrm{diag}\left(\Lambda_{\mathrm{eff}}\boldsymbol{\alpha}_{\mathrm{eff}}\right)B_{\mathrm{eff}}$$

$$= \left\{ \left(1 + ib\omega_j\right)A \quad \left(-\omega_j^2 + ia\omega_j\right)A_m \right\} \mathrm{diag}\left(\begin{Bmatrix} \Lambda & 0 \\ 0 & \Lambda_m \end{Bmatrix} \begin{Bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\alpha}_m \end{Bmatrix} \right) \begin{Bmatrix} A^T \\ A_m^T \end{Bmatrix}$$

Iterative Enclosure Method

- Complete governing equation, after decomposition:

$$\begin{Bmatrix} K_{\mathrm{eff},j0} & C^T \\ C & 0 \end{Bmatrix} \begin{Bmatrix} \mathscr{F}_t(\mathbf{u})_j \\ \mathscr{F}_t(\boldsymbol{\lambda})_j \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \boldsymbol{\delta}\mathscr{F}_t(\mathbf{d}_t)_j - \begin{Bmatrix} A_{\mathrm{eff}} \\ 0 \end{Bmatrix} \mathrm{diag}\left(B_{\mathrm{eff}}\mathscr{F}_t(\mathbf{u})_j\right)\Lambda_{\mathrm{eff}}\Delta\boldsymbol{\alpha}_{\mathrm{eff}}$$

- Note: fixed-point form

Iterative Enclosure Method

- Rewrite into the following compact form

$$K_{g,j}\mathscr{F}_t(\mathbf{u}_g)_j = F_g\boldsymbol{\delta}\mathscr{F}_t(\mathbf{d}_t)_j - A_{g,j}\mathrm{diag}\left(B_{\mathrm{eff}}\mathscr{F}_t(\mathbf{u})_j\right)\Lambda_{\mathrm{eff}}\Delta\boldsymbol{\alpha}_{\mathrm{eff}}$$

- Apply IDFT

$$\mathbf{u}_{g,k} = \left(\mathscr{F}_t^{-1}(G_jF_g)*\mathbf{d}_t\right)_k\boldsymbol{\delta} - \left(\mathscr{F}_t^{-1}(G_jA_{g,j})*\mathrm{diag}\left(B_{\mathrm{eff}}\mathbf{u}\right)\right)_k\Lambda_{\mathrm{eff}}\Delta\boldsymbol{\alpha}_{\mathrm{eff}}$$

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Example 1 - Rigid Frame

- Damping:

$$\mathbf{C} = 20\mathbf{M} + 3\times10^{-5}\mathbf{K},$$

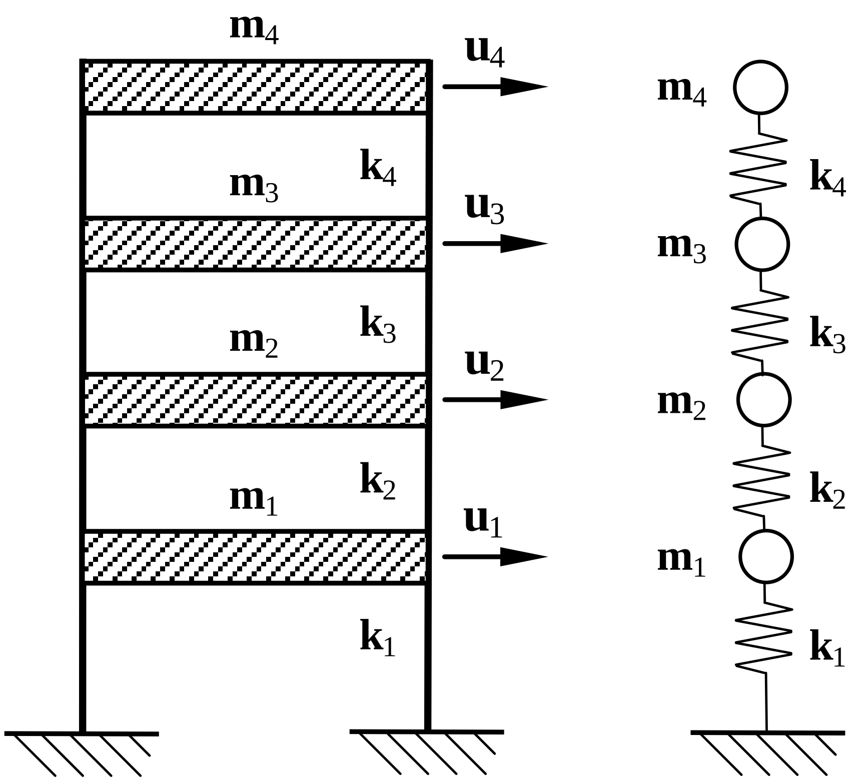
- Sampling interval:

$$\Delta t = 1\times10^{-4}\ \text{s}$$

- Initial conditions:

$$\mathbf{u}_4 = [0.99, 1.01]\times10^{-3}\ \mathbf{m}$$

$$\mathbf{v}_4 = [0.99, 1.01]\times10^{-2}\ \mathbf{m/s}$$



Example 1 - Rigid Frame

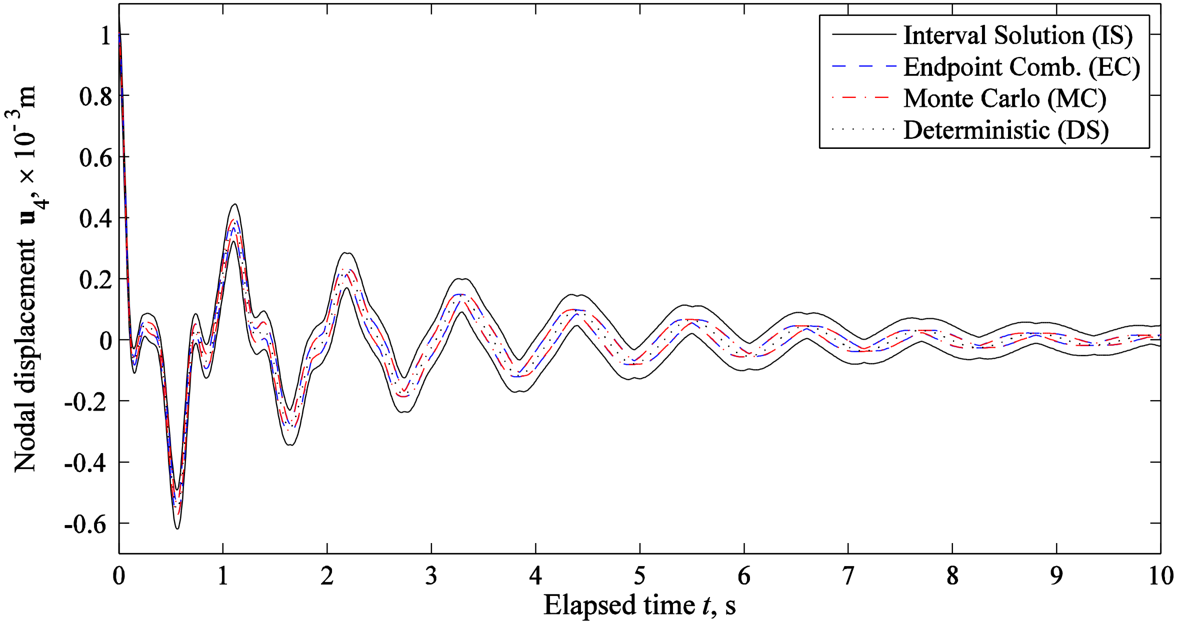
- Uncertainties in mass and stiffness

Table I. Interval mass and stiffness for the five-story rigid frame of Figure 1, including 1% uncertainties in mass, and 5% uncertainties in stiffness.

Floor	Mass (kg)			Stiffness (kN/m)		
	$\mathbf{m}_j$	mid $\mathbf{m}_j$	rad $\mathbf{m}_j$	$\mathbf{k}_j$	mid $\mathbf{k}_j$	rad $\mathbf{k}_j$
1	[5.416, 5.470]	5.443	0.027	[1.180, 1.240]	1.210	0.030
2	[5.416, 5.470]	5.443	0.027	[1.677, 1.763]	1.720	0.043
3	[5.416, 5.470]	5.443	0.027	[1.862, 1.958]	1.910	0.048
4	[5.416, 5.470]	5.443	0.027	[1.775, 1.865]	1.820	0.045

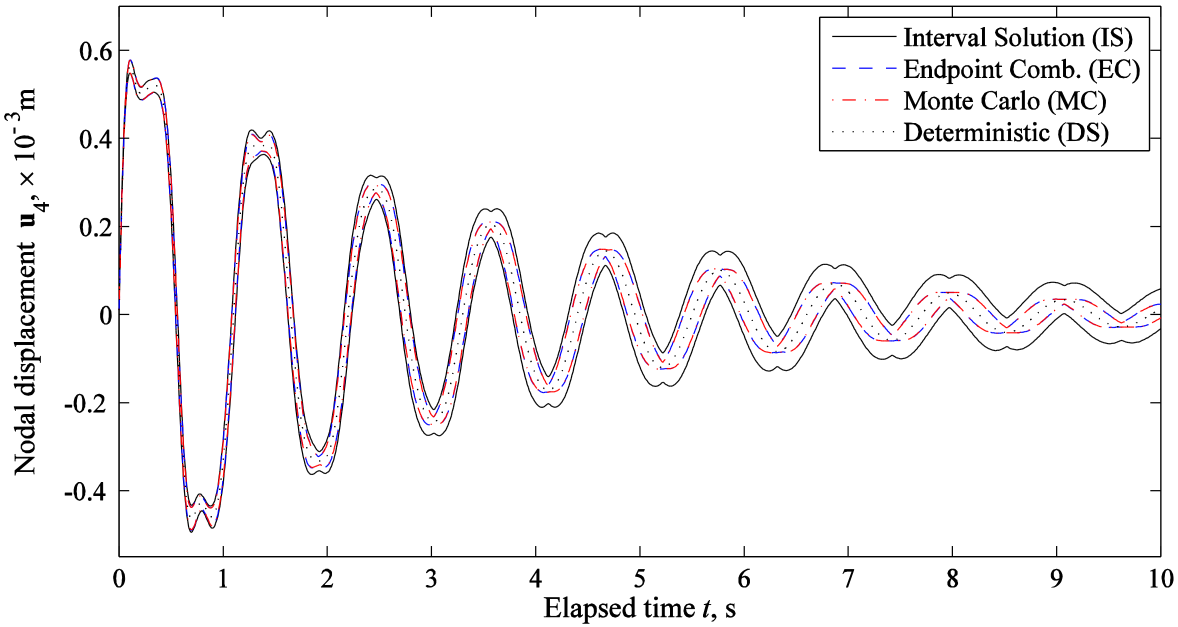
Example 1 - Rigid Frame

- Dynamic response due to non-zero initial displacement at the top of the frame



Example 1 - Rigid Frame

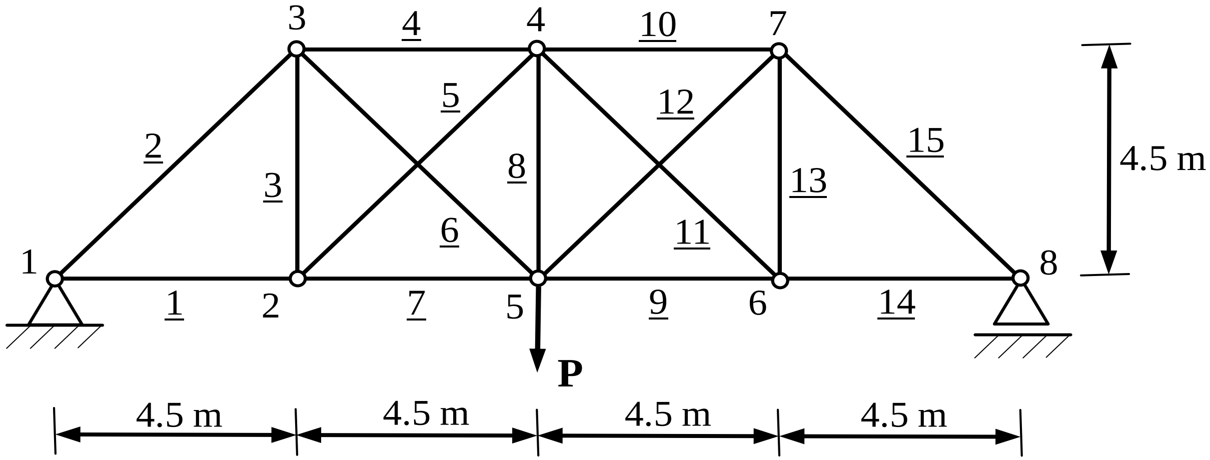
- Dynamic response due to non-zero initial velocity at the top of the frame



Example 2 - Simple Truss

- Bars 4-12:

  $A = 6 \times 10^{-4}\ \mathrm{m}^{-2}$



- Others:

  $A = 1 \times 10^{-3}\ \mathrm{m}^{-2}$

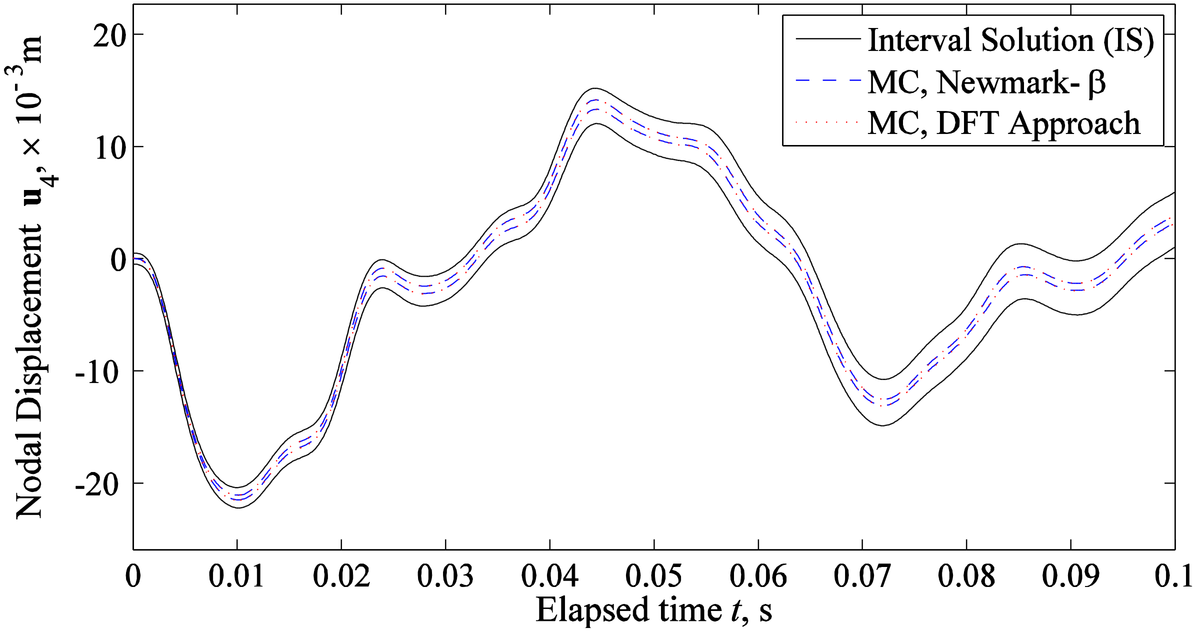
- Midpoint values: $E = 200\,\mathrm{GPa}, \quad \rho = 7.8 \times 10^{3}\ \mathrm{kg/m}^{3}$

  $P = 200 \sin(100\pi t)\ \mathrm{kN}$

- Damping: $\mathbf{C} = 20\mathbf{M} + 3 \times 10^{-5}\mathbf{K}, \quad \Delta t = 1 \times 10^{-4}\ \mathrm{s}$

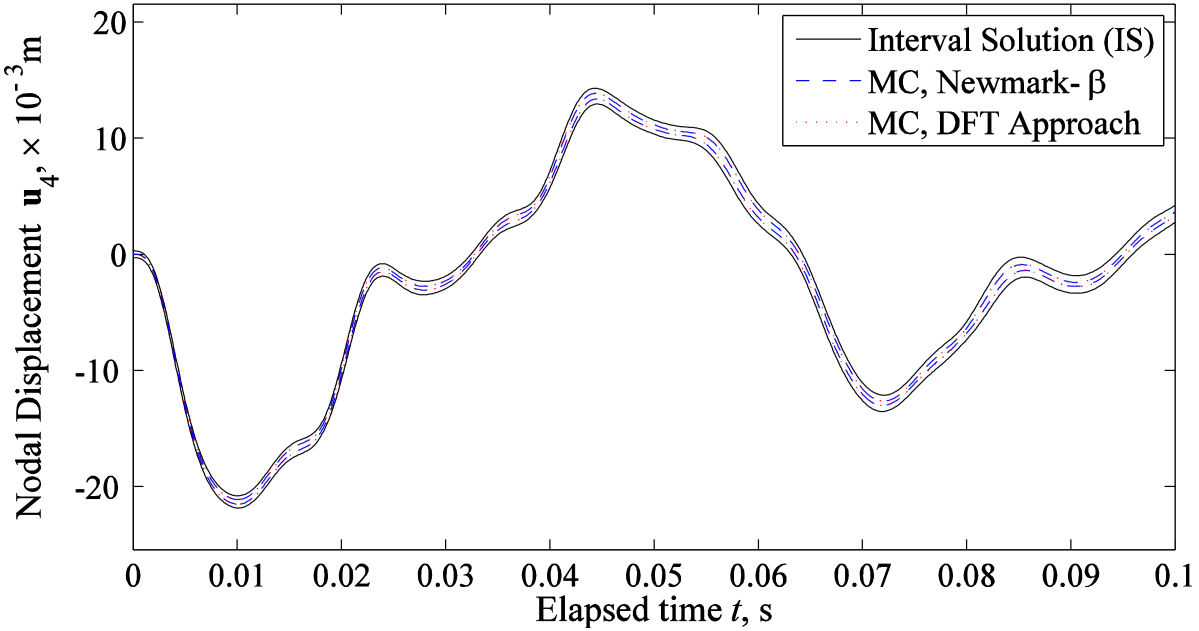
Example 2 - Simple Truss

- 1% uncertainty in mass, stiffness, load magnitude and load history



Example 2 - Simple Truss

- 2% uncertainty in mass, stiffness, and load magnitude (not in load history)



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Conclusions

- Structural dynamic response under interval uncertainty
- The use of DFT (and IDFT) to reduce overestimation
- Guaranteed enclosure
- Tight bounds due to matrix decomposition and iterative enclosure method

THANK YOU!