Why Min-Based Conditioning

Salem Benferhat[1] and Vladik Kreinovich[2]

[1]CRIL (Centre de Recherche en Informatique de Lens)
CNRS – UMR 8188
Université d'Artois, Faculté des sciences Jean Perrin
Rue Jean Souvraz, SP 18, F62307 Lens Cedex, France
benferhat@cril.univ-artois.fr

[2]Department of Computer Science
University of Texas at El Paso
El Paso, Texas 79968, USA
vladik@utep.edu

Need for Ordinal-Scale . . .
From Ordinal Scale to . . .
Need for Conditioning . . .
Reasonable Properties
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Final Property: Invariance
Main Result
Proof

1. Need for Ordinal-Scale Possibility Degrees

- It is often useful to describe,
  - for each theoretically possible alternative ω from the set of all *theoretically* possible alternatives Ω,
  - to what extent this alternative is, in the expert's opinion, *actually* possible.
- Often, the only information that we can extract from experts is the qualitative one:
  - which alternatives have a higher degree of possibility and
  - which have lower degree.
- In some cases, we have a linear order.
- We could use this order to process this information.
- However, computers have been designed to process numbers; they are still best in processing numbers.

2. From Ordinal Scale to Numbers

- So, degrees of possibility are usually described by numbers $\pi(\omega) \in [0,1]$:
  - the higher the degree,
  - the larger the value $\pi(\omega)$.
- These numbers by themselves do not have an exact meaning, the only meaning is in the order.
- So, the same meaning can be described if we apply any strictly increasing transformation to $[0,1]$.
- Usually, some of this freedom is eliminated by the convention that the largest degree is set to 1.
- We can always achieve this with an appropriate transformation (*normalization*).
- **Definition.** *Let* Ω *be a finite set. A* possibility distribution *is a function* $\pi : \Omega \to [0,1]$ *s.t.* $\max\limits_{\omega\in\Omega} \pi(\omega) = 1$.

3. Need for Conditioning and Normalization

- Often, we acquire an additional information:
  - some of the alternatives that we originally thought to be possible
  - are actually not possible: $\Psi \subset \Omega$, $\Psi \neq \Omega$.
- *Example:* some original suspects have alibis.
- We have $\pi'(\omega) = 0$ for all $\omega \notin \Psi$; but we may have $\max\limits_{\omega \in \Psi} \pi'(\omega) < 1$, so we need normalization.
- **Definition.** *By a* conditioning operator, *we mean a mapping* $(\pi \,|\, \Psi)$ *that:*
  - *inputs a possibility distribution* π *on a set* Ω *and a non-empty set* $\Psi \subseteq \Omega$*, and*
  - *returns a new possibility distribution for which* $(\pi \,|\, \Psi)(\omega) = 0$ *for all* $\omega \notin \Psi$*.*
- What are the reasonable conditioning operators?

4. Reasonable Properties

- A first reasonable requirement is that:
  - since alternatives $\omega \notin \Psi$ are excluded,
  - their original possibility degrees should not affect the resulting degrees.
- **C1.** If $\pi_{|\Psi} = \pi'_{|\Psi}$, i.e., if $\pi(\omega) = \pi'(\omega)$ for all $\omega \in \Psi$, then $(\pi \mid \Psi) = (\pi' \mid \Psi)$.
- Another reasonable condition is that:
  - while the numerical values of possibility degrees may change,
  - the order between these degrees should not change.
- **C2.** If $\pi(\omega) < \pi(\omega')$ for some $\omega, \omega' \in \Psi$, then $(\pi \mid \Psi)(\omega) < (\pi \mid \Psi)(\omega')$.
- **C3.** If $\pi(\omega) = \pi(\omega')$ for some $\omega, \omega' \in \Psi$, then $(\pi \mid \Psi)(\omega) = (\pi \mid \Psi)(\omega')$.

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5. Reasonable Properties (cont-d)

- Often, after learning $\Psi \subset \Omega$, we learn additional information $\Psi' \subset \Psi$. In this case:
  - first compute $\pi' = (\pi \,|\, \Psi)$, and then
  - compute $\pi'' = (\pi' \,|\, \Psi') = ((\pi \,|\, \Psi) \,|\, \Psi')$.
- Alternative, we could learn both pieces of the information at the same time, and get $(\pi \,|\, \Psi')$.
- In both cases, we gain the exact same new information.
- So, the resulting changes in possibility degrees should be the same:
- **C4.** If $\Psi' \subset \Psi$, then $((\pi \,|\, \Psi) \,|\, \Psi') = (\pi \,|\, \Psi')$.

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6. Reasonable Properties (cont-d)

- Another condition is that if had an alternative ω_0 which we originally believed to be impossible, then:
  - this alternative should remain impossible, and
  - the possibility degrees of all other alternatives $\omega \neq \omega_0$ should remain the same.
- **C5.** If $\pi(\omega_0) = 0$ for some $\omega_0 \in \Psi$, then $(\pi \mid \Psi)(\omega_0) = 0$ and $(\pi_{|\Psi - \{\omega_0\}} \mid \Psi) = (\pi \mid \Psi)_{|\Psi - \{\omega_0\}}$.

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7. Final Property: Invariance

- What matters is the order between the degrees, not the numerical values of the degrees.
- So, the situations should not change if we apply a re-scaling T that doesn't change the order (e.g., $x \to x^2$).
- The result of applying the conditioning operator not change if we apply such a re-scaling.
- We should get the exact same result:
  - if we apply conditioning $\pi \to (\pi \,|\, \Psi)$ in the original scale, and then re-scale to $T(\pi \,|\, \Psi)$;
  - or we first apply the re-scaling, resulting in $T\pi$, and then apply the conditioning, resulting in $(T\pi \,|\, \Psi)$.
- **C6.** For every increasing one-to-one function $T : [0, 1] \to [0, 1]$, we have $(T\pi \,|\, \Psi) = T(\pi \,|\, \Psi)$.

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8. Main Result

- *Proposition. The only conditioning operator satisfying* **C1**–**C6** *is the* min-based operator *for which:*
  - $(\pi \,|\, \Psi)(\omega) = 1$ *when* $\omega \in \Omega$ *and* $\pi(\omega) = \max\limits_{\omega' \in \Psi} \pi(\omega')$;
  - $(\pi \,|\, \Psi)(\omega) = \pi(\omega)$ *when* $\omega \in \Omega$ *and* $\pi(\omega) < \max\limits_{\omega' \in \Psi} \pi(\omega')$; *and*
  - $(\pi \,|\, \Psi)(\omega) = 0$ *when* $\omega \notin \Psi$.
- The usual derivation selects $(A \,|\, B)$ as the maximal value s.t. $d(A \,\&\, B) = d((A \,|\, B) \,\&\, B)$, with

$$d(A \,\&\, B) \stackrel{\text{def}}{=} \min(d(A), d(B)).$$

- We show that *maximality* can be replaced with *invariance* – reflecting ordinal character of degrees.

9. Proof

- It is easy to show that the min-based operator satisfies the properties **C1–C6**.
- To complete the proof, we need to prove that, vice versa,
  - every conditioning operator that satisfies these five properties
  - is indeed the min-based operator.
- To prove this statement, we will consider two possible cases:
  - the case when the set Ψ contains some alternative ω for which $\pi(\omega) = 1$, and
  - the case when the set Ψ does not contain any alternative ω for which $\pi(\omega) = 1$.

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10. Proof: First Case

- Let us first consider the case when the set Ψ contains some alternative ω for which $\pi(\omega) = 1$.
- In this case, the min-based formula leads to $(\pi \mid \Psi)(\omega) = \pi(\omega)$ for all $\omega \in \Psi$.
- Let us show that this equality holds for all conditioning operators that satisfy the properties **C1**–**C6**.
- If there is no $\omega_0 \in \Psi$ for which $\pi(\omega_0) = 0$, let us add such an element to our set Ω.
- According to Property **C5**, this will not change the result.
- Thus, without losing generality, we can safely assume that there is an element $\omega_0 \in \Psi$ for which $\pi(\omega_0) = 0$.
- As for the values $\pi(\omega)$ for $\omega \notin \Psi$, we can use the property **C1** to replace them with zeros.

11. First Case (cont-d)

- Let us sort values $\psi(\omega)$ corresponding to different alternatives $\omega \in \Psi$ in increasing order.
- We know that the resulting list of values includes 0 and 1, so this list has the form

$$v_1 = 0 < v_2 < \ldots < v_{k-1} < v_k = 1.$$

- Let us use property **C6** to prove that the values $(\pi \mid \Psi)$ should also be from this list.
- Indeed, let us consider the following strictly increasing function $T(v)$: for $v_i \le v \le v_{i+1}$, we take

$$T(v) = v_i + \left(\frac{v - v_i}{v_{i+1} - v_i}\right)^2 \cdot (v_{i+1} - v_i).$$

- One can easily check that for this function, $T(v_i) = v_i$ for all i, so $T(\pi) = \pi$.

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12. First Case (cont-d)

- Thus, the property **C6** implies that $T(\pi \mid \Psi) = (\pi \mid \Psi)$.
- So, for each value $v = (\pi \mid \Psi)(\omega)$, we should have $T(v) = v$.
- But for the above function $T(v)$, the only such values are $v_1, \ldots, v_k$.
- So, indeed, the values $v_1 < \ldots < v_k$ are mapped to the same k values.
- By properties **C2** and **C3**:
  - equal values of $\pi(\omega)$ are mapped into equal values of $(\pi \mid \Psi)(\omega)$, and
  - smaller values of $\pi(\omega)$ are mapped into smaller values of $(\pi \mid \Psi)(\omega)$.
- Thus, the values v'_i corresponding to v_i are also sorted in increasing order: $v'_1 < \ldots < v'_k$.

13. First Case (final)

- Each new value v'_i must coincide with one of the original values v_j.
- So, in the increasing list $v_1 < \ldots < v_k$ of k values, we have k new values v'_i which have the same order.
- This implies:
  - that v'_1 must be the smallest of v_i, i.e., $v'_1 = v_1$,
  - that v'_2 be the second smallest, i.e., $v'_2 = v_2$, and,
  - in general, $v'_i = v_i$.
- So, indeed, $(\pi \,|\, \Psi)(\omega) = \pi(\omega)$ for all $\omega \in \Psi$.

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14. Proof: Second Case

- Let us now consider the case when the set Ψ does not contain some alternative ω for which $\pi(\omega) = 1$.
- In this case, we can also:
  - add (if needed) an element ω_0 for which $\pi(\omega_0) = 0$, and
  - sort the values $\pi(\omega)$ corresponding to $\omega \in \Psi$ into an increasing sequence $v_1 = 0 < v_2 < \ldots < v_k < 1$.
- The only difference is that in this case, the largest value v_k in this increasing sequence is smaller than 1.
- One of the new values should be equal to 1.
- So, due to Properties **C2** and **C3**, only the largest degree v_k should be mapped into 1.

15. Second Case (cont-d)

- Similarly to the first case, we can prove:
  - that each of the the values $v_1, \ldots, v_{k-1}$ maps into one of the values $v_1, \ldots, v_k$, and
  - that if $v_i < v_j$, then $v'_i < v'_j$.
- By induction, we can prove that $v'_i \geq v_i$.
- Since we have only one additional value to move to, for each i, we have either $c'_i = v_i$ or $v'_i = v_{i+1}$.
- Let use the Property **C4** to prove, by contradiction, that $v_i < v_k$ cannot be transformed into v_{i+1}.
- Let us assume that, vice versa, there is an element $\omega_i \in \Psi$ for which $\pi(\omega_i) = v_i$ and $(\pi \,|\, \Omega)(\omega_i) = v_{i+1}$.

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16. Second Case (cont-d)

- To get a contradiction, let us consider:
  - the new set $\Omega^* = \Omega \cup \{\omega^*\}$, with a new element ω^*, and
  - a new possibility distribution π^* for which we have $v_i < \pi^*(\omega^*) < v_{i+1}$ and $\pi^*(\omega) = \pi(\omega)$ for all $\omega \neq \omega_i$.
- Let us consider two conditionining paths from Ω^* to Ψ:
  - in the first path, we go from Ω^* to Ω and then from Ω to Ψ;
  - in the second path, we go from Ω^* to $\Psi^* \stackrel{\text{def}}{=} \Psi \cup \{\omega^*\}$ and then from Ψ^* to Ψ.
- According to the Property **C4**, the resulting value $(\pi^* \mid \Psi)(\omega_i)$ should be the same for both paths.
- In the first path, first, we go from Ω^* to Ω.

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17. Second Case (cont-d)

- The transition from Ω^* to Ω eliminates a single element ω^* for which $\pi^*(\omega^*) < 1$.
- Thus, according to the first case, possibility degrees of remaining elements remain unchanged: $(\pi^* \mid \Omega) = \pi$.
- We already know that $(\pi \mid \Psi)(\omega_i) = v_{i+1}$.
- Thus, due to Property **C4**, we have

$$(\pi^* \mid \Psi)(\omega_i) = ((\pi^* \mid \Omega) \mid \Psi)(\omega_i) = v_{i+1}.$$

- On the other hand, in the second path, we first move from Ω^* to Ψ^*.
- In this transition, we have v_k transformed into 1, and the original value $\pi^*(\omega_i) = v_i$:
  - can either remain the same,
  - or it can be transformed to the next value which is now $\pi^*(\omega^*) < v_{i+1}$.

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18. Second Case (cont-d)

- In both cases, the new possibility degree is smaller than v_{i+1}: $\pi(\omega_i) < v_{i+1}$.
- When we then reduce Ψ^* to Ψ, then:
  - all alternatives for which originally $\pi^*(\omega) = \pi(\omega) = v_k$ and now $\pi'(\omega) = 1$
  - remain in the set.
- Thus, all other alternatives – including the alternative ω_i – according to first case, retain their values.
- For ω_i, this implies that $(\pi' \mid \Psi)(\omega_i) = \pi'(\omega_i) < v_{i+1}$.
- Thus, we have $(\pi^* \mid \Psi)(\omega_i) = \pi'(\omega_i) < v_{i+1}$.
- This contradicts to $(\pi^* \mid \Psi)(\omega_i) = v_{i+1}$.
- This contradiction shows that the transformation from v_i to v_{i+1} is indeed impossible. So, $v_i' = v_i$. Q.E.D.

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