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Model-Order Reduction Using Interval Constraint Solving Techniques

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Dynamical systems

are a principal tool in modeling and control of many physical phenomena:

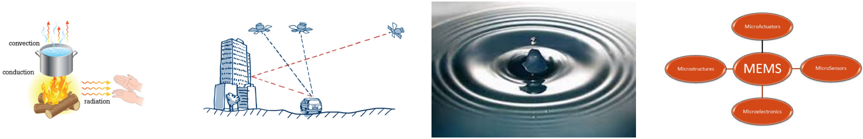


Figure: heat transfer, signal propagation, wave propagation, and mems systems

$$\{\text{need for accuracy} \longrightarrow \text{more details in the modeling}\} \longrightarrow \text{larger dynamical systems}$$

$$\text{larger dynamical systems} \longrightarrow \text{unmanageably large demands on computational resources}$$

MODEL-ORDER REDUCTION

Given the function

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

a nonlinear system of equations consists in finding x such that:

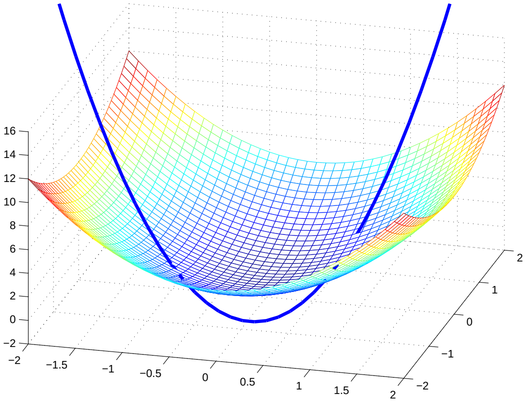
$$F(x) = 0$$

MOR is typically performed on the premise that the solution x belongs to an affine subspace, W, of $\mathbb{R}^n$ whose dimension k is orders of magnitude smaller than n, i.e.

$$x = z + \Phi p$$

where Φ is a basis of a subspace of $\mathbb{R}^n$ associated to W

NONLINEAR SYSTEM OF EQUATIONS. EXAMPLE

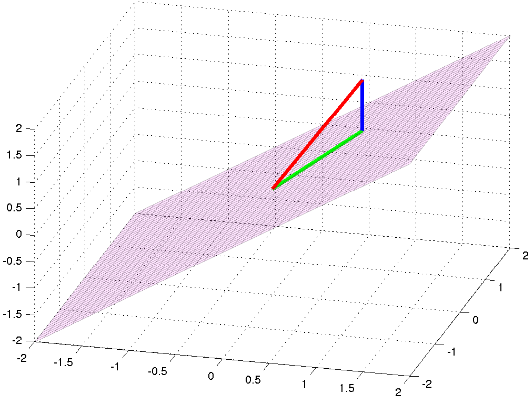


$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$F(X) = 0$$

$$\begin{aligned} 2x_1^2 + x_2^2 - x_3 &= 0 \\ x_1 - x_2 &= 0 \\ 10x_1^2 - x_3 - 4 &= 0 \end{aligned}$$

ILLUSTRATION OF MODEL-ORDER REDUCTION



$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\begin{aligned} F(X) &= 0 \\ F(\Phi p + z) &= 0 \end{aligned}$$

Assuming

$$z \approx 0$$

we have to solve:

$$F(\Phi p) = 0$$

$$p = \arg\min \frac{1}{2} ||F(\Phi p)||^2$$

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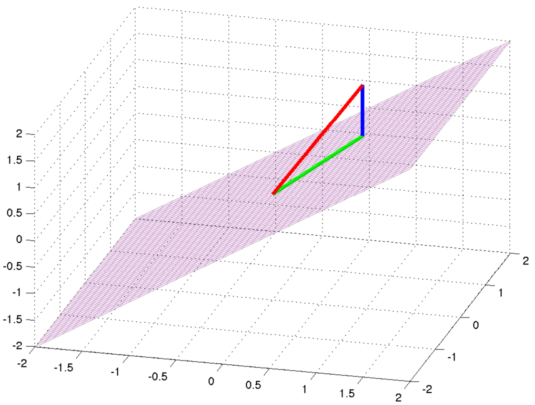

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \\ \phi_{31} & \phi_{32} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

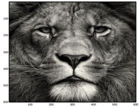
MODEL-ORDER REDUCTION. BASICS

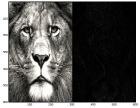
- The goal of Model Order Reduction (MOR) is to:
  - Reduce complexity
  - Maintain (input-output) accuracy
  - Maintain relevant physical properties
- A good Reduction methodology must be:
  - Accurate, efficient, numerically robust, and generate useful models
- MOR can be applied in many different settings: e.g.,
  - Linear Systems
  - Parametric Systems
  - Non-Linear Systems

MODEL-ORDER REDUCTION TECHNIQUES

There are several techniques of MOR, but all of them have the same goal: finding a basis Φ. For example:

1. Krylov method: $\Phi = \mathcal{K}_k(A, b) = \{b,\ Ab\ , A^2 b, \cdots, A^{k-1} b\}$
2. Wavelets method: Let $W = \begin{pmatrix} L \\ H \end{pmatrix}$ a discrete wavelet: $\Phi = L^T$





3. Proper Orthogonal Decomposition (POD): Based on Principal Component Analysis (PCA), which is a procedure for identifying a smaller number of uncorrelated variables, called "principal components", from a large set of data. The goal of PCA is to describe the maximum amount of variance with the fewest number of principal components.

ILLUSTRATION OF THE POD METHOD

Given the parametric nonlinear system of equations

$$R(x, \lambda) = 0$$

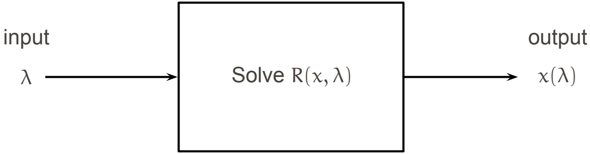


ILLUSTRATION OF THE POD METHOD: SNAPSHOTS

Collect snapshots, i.e.

1. Solve $R(x, \lambda) = 0$ for different λ values (input)
2. Save the snapshots, i.e. the solution $x(\lambda)$.
3. Organize them in a “snapshots” matrix

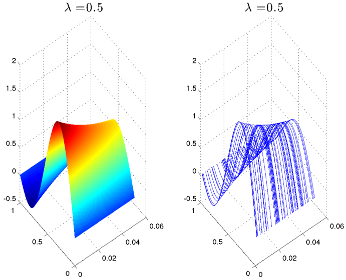


ILLUSTRATION OF THE POD METHOD: FINDING Φ

Find basis Φ

1. Compute the Singular Value Decomposition (SVD) of the snapshots matrix

$$[U, S, V] = svd(snapshots)$$

2. Take $\Phi = U(:, [1:k])$, that is, take the first k columns of U such that:

$$\frac{\sum_{i=1}^{k} S(i)}{\sum_{i=1}^{n} S(i)} > \varepsilon$$

USING Φ TO REDUCE THE PROBLEM

For a fixed λ, we solve

$$F(x) = R(x, \lambda) = 0$$

For Full-Order Model, we use:

Newton's method

- Set $i = 0$
- Guess an approximation of the solution u^0
- **Repeat**
  - Compute $J(u^i)$, Jacobian of F, and $F(u^i)$
  - Solve the linear system $J(u^i)\Delta u = -F(u^i)$
  - Set $u^{i+1} = u^i + \Delta u$
  - Set $i = i + 1$
- **Until convergence**

For Reduced-Order Model, we use:

Reduced Newton's method

- Set $i = 0$
- Guess an approximation of the solution p^0
- **Repeat**
  - Compute $J(\Phi p^i)$ and $F(\Phi p^i)$
  - Solve the linear system $J(\Phi p^i)\phi\Delta p = -F(\Phi p^i)$
  - Set $p^{i+1} = p^i + \Delta p$
  - Set $i = i + 1$
- **Until convergence**

DISADVANTAGES OF POD

Although POD is one of the most popular approaches to MOR, it presents several disadvantages:

1. POD requires a series of offline computations in order to form the matrix of snapshots.
2. The quality of the resulting reduced basis heavily depends on the choice of parameters and inputs.
3. The accuracy of these over which the snapshots are computed is an issue.

In this work, we propose **an Interval version of POD**.

COMPUTATIONS WITH INTERVALS

In this work, when mentioning intervals, we actually mean *closed real-value-bounded intervals*, so an interval X is defined:

$$X = [\underline{X}, \overline{X}] = \{x \in \mathbb{R} : \underline{X} \leq x \leq \overline{X}\}$$

We are going to manipulate intervals, for instance:

- **Addition:** $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$
- **Substraction:** $X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]$
- **Multiplication:** $X \cdot Y = \{\min S, \max S\}$, where $S = \{\underline{X} \cdot \underline{Y}, \underline{X} \cdot \overline{Y}, \overline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}\}$
- **Division:** $X/Y = \{\min S, \max S\}$, where $S = \{\underline{X}/\underline{Y}, \underline{X}/\overline{Y}, \overline{X}/\underline{Y}, \overline{X}/\overline{Y}\}$, if $0 \notin Y$

In general:

$$X \Diamond Y = \Box\{x \Diamond y, \text{ where } x \in X \text{ and } y \in Y\}$$

where $\Diamond$ stands for any arithmetic operator, including division where $0 \in Y$, and $\Box$ is the hull operator.

HOW TO SOLVE NONLINEAR EQUATIONS WITH INTERVALS?

Branch-and-Bound

It is the underlying principle of search in interval constraint solving techniques, and allows to guarantee completeness of the search.

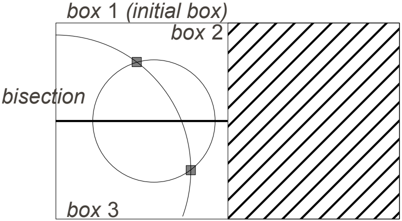


Figure taken from Laurent Granvilliers, RealPaver User's Manual.

Algorithm

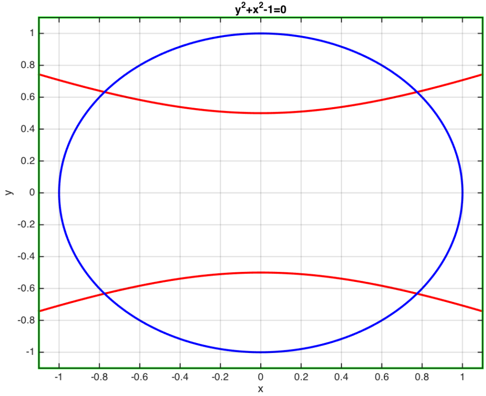
Input: System of constraints $C = \{c_1, \ldots, c_k\}$, a search space D_0.
Output: A set Sol of interval solutions

Set Sol to empty
If $\forall i$, $0 \in F_i(D_0)$ then:
    Store D_0 in some storage S
    While (S is not empty) do:
        Take D out of S
        If ($\forall i$, $0 \in F_i(D)$) then:
            If (D is still too large) then:
                Split D in D_1 and D_2
                Store D_1 and D_2 in S
            Else:
                Store D in Sol
Return Sol

EXAMPLE OF BRANCH AND BOUND

Let us suppose we want to solve the nonlinear system of equations:

$$\begin{cases} (2y)^2 - x^2 & = & 1 \\ y^2 + x^2 & = & 1 \end{cases}$$



INTERVAL ARITHMETIC + POD

Let us recall the problem we are solving

$$R(x,\lambda) = 0, \qquad \lambda \in \mathbf{I}$$

POD method

$$\begin{aligned} R(x,\lambda_1) &= 0,\\ R(x,\lambda_2) &= 0,\\ \vdots \quad & \quad \vdots \\ R(x,\lambda_n) &= 0, \end{aligned}$$

where $\lambda_i \in \mathbf{I}$, for $i = 1, 2, \ldots, n$

What do we propose?

we propose an **interval version of POD**

INTERVAL POD (I-POD)

Let us recall the problem we are solving

$$R(x, \lambda) = 0, \qquad \lambda \in \mathbf{I}$$

POD method

$$\begin{aligned} R(x, \lambda_1) &= 0, \\ R(x, \lambda_2) &= 0, \\ \vdots & \quad \vdots \\ R(x, \lambda_n) &= 0, \end{aligned}$$

where $\lambda_i \in \mathbf{I}$, for $i = 1, 2, \ldots, n$

I-POD method

we propose an **interval version of POD**

$$R(x, \mathbf{I}) = 0$$

NUMERICAL RESULTS

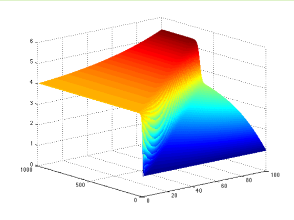
Consider the Burgers' equation:

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial f(U(x,t))}{\partial x} = g(x), \qquad (1)$$

where U is the unknown conserved quantity (mass, density, heat etc.), $f(U) = 0.5U^2$ and in this example, $g(x) = 0.02\,\mathtt{exp}(0.02x)$. The initial and boundary conditions used with the above PDE are: $U(x;0) \equiv 1$; $U(0;t) = \lambda$, for all $x \in [0;100]$, and $t > 0$.

POD method

We solve the Burgers' equation for $\lambda_i \in [3.5, 4.5]$



Consider the Burgers' equation:

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial f(U(x,t))}{\partial x} = g(x), \qquad (2)$$

where U is the unknown conserved quantity (mass, density, heat etc.), $f(U) = 0.5U^2$ and in this example, $g(x) = 0.02\,\mathtt{exp}(0.02x)$. The initial and boundary conditions used with the above PDE are: $U(x;0) \equiv 1$; $U(0;t) = \lambda$, for all $x \in [0;100]$, and $t > 0$.

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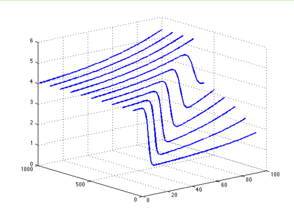

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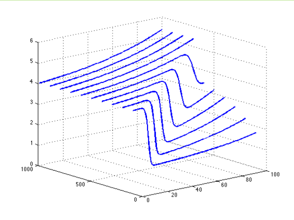

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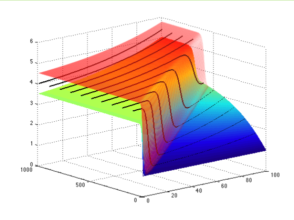

NUMERICAL RESULTS

Method	Tag 1	Tag 2	Tag 3	Tag 4
POD	300 secs	37	0.75 secs	$4.85\mathrm{E}-4$
I-POD	94.45	36	0.75 secs	$5.76\mathrm{E}-4$

Tag 1: Time computing the Reduced basis

Tag 2: Dimension of the Subspace

Tag 3: Time solving Burgers' equation using the obtained basis

Tag 4: $||u_{\mathrm{fom}} - u_{\mathrm{rom}}|| / ||u_{\mathrm{fom}}||$

ADVANTAGES AND LIMITATIONS OF IPOD

The major two advantages of our proposed method are:

- the computational time it requires to obtain the snapshots: Our approach requires less time than the original one and the quality of the snapshots our method generates is comparable to that generated by POD; and
- the ability to handle uncertainty: the interval that contains λ, handled at once by IPOD, is similar to uncertainty and is handled without problems.

The major limitation of IPOD is that exaggerates the overestimation when the dynamic system is highly nonlinear.

CONCLUSIONS AND FUTURE WORK

Conclusions:

1. We proposed and described a novel Model-Order Reduction approach that improves the well-known Proper Orthogonal Decomposition method (POD) by using Interval analysis and Interval Constraint Solving Techniques.
2. We observed and reported promising performance of IPOD, when compared to POD.

Future Work

1. We do plan to challenge IPOD on problems whose solution is highly nonlinear, e.g., the Fitz-Hugh-Nagumo (FHN) problem.
2. We need to assess its relevance in handling and meaningfully solving problems with other sources of uncertainty; e.g., uncertainty in the initial condition.
3. When having to handle uncertainty, achieving a relevant reduced basis is not all that needs to be modified from traditional approaches: once the space reduced, solving techniques (currently, namely, Newton-based methods) need to be extended to intervals.

ACKNOWLEDGMENT

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