Static Analysis of Structural Systems with Uncertain Parameters Using Probability-Boxes

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REC

Imprimers

Ted Belytschko
Ivo Babushka
Vladik Kreinovich
Ray Moore
A. Neumier
Scott Ferson

So, three mathematicians go into a bar. The bartender asks

"Do you ALL want beers?"

Conventional probability

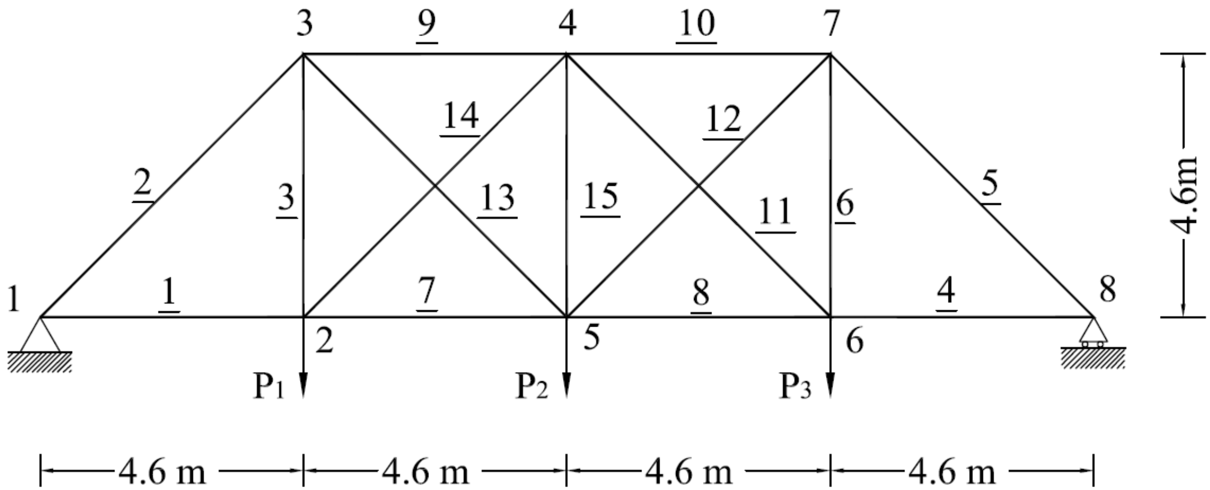
- Each person wants a beer but doesn't know about the others. Following conventional probability and uninformed prior
- 1*.5*.5 = .25
- Each answers Probably not.

Admitting what you don't know

- First person answers: I don't know
- Second person answers: I don't know
  - Third person's only logical answer is:

YES!

Motivation : A toy problem.



A truss structure (extendable to general FEA)
Loading (and parameters) given in terms of random variables (RV)

p = f (P1, P2, P3) vs p= f1(P1), p=f2(P2), p=f3(P3)

What can one compute about structural response when dependency between p1, p2, p3 is unknown?

Common Engineering Solution:

Assume independence between P1, P2, P3 use Monte Carlo simulation and forget about it.

Common Engineering Solution:

Assume independence between P1, P2, P3 and forget about it.

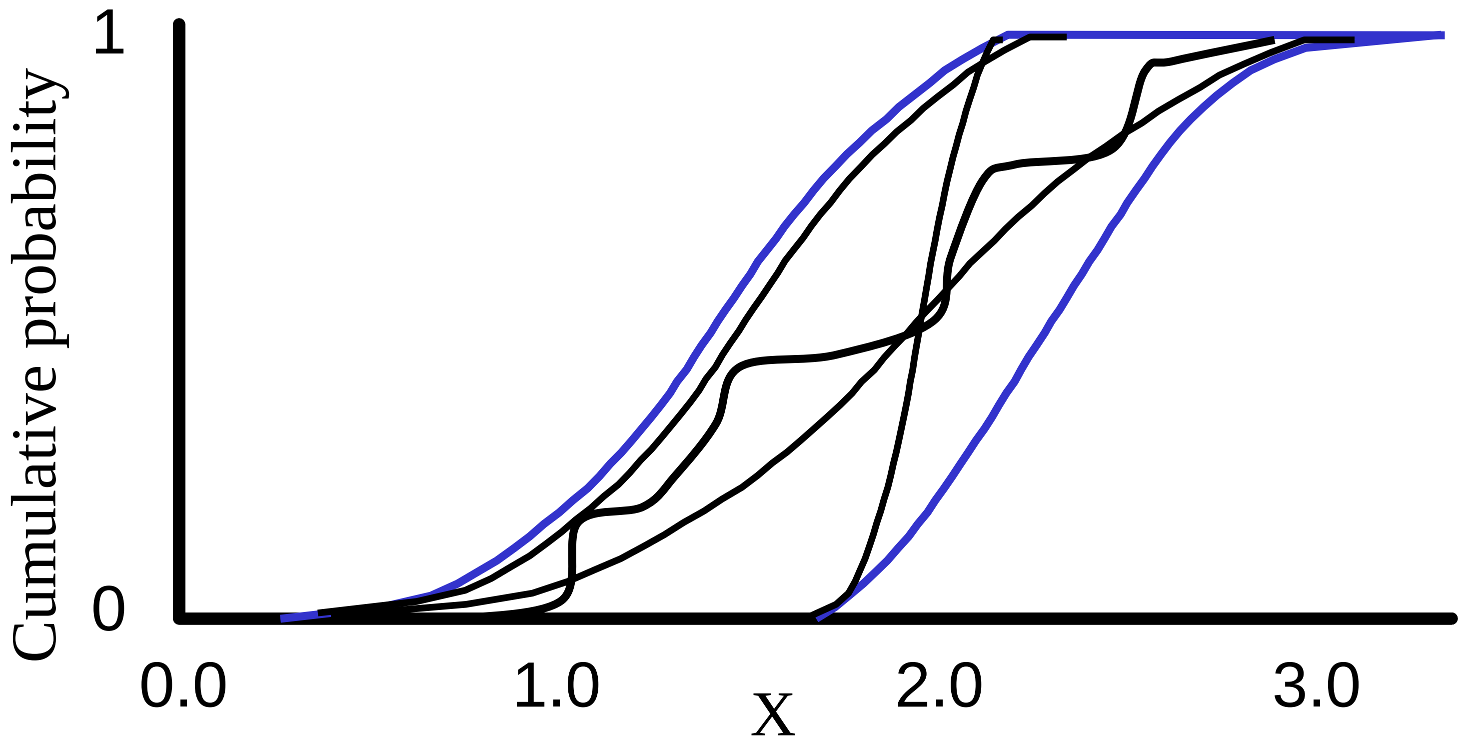
Assuming independence can result in significant underestimation of risk (Ferson et al. 2004).

Outline

- Introduction
- Probability bounds and Structural Analyses
  - P box for representing ignorance
  - Any dependency bounds
  - Nonlinear problems.
- Development of solution of matrix equations with coefficient matrix defined by P-boxes
- Results
- Conclusions

Topic	REC2004	REC 2016
Engineering Analysis	Truss structural analysis	General FEM
Boundary conditions	Interval loads	P-box loads
Parameters	Interval Parameters 1	P-box parameters
Derived Quantities		Same as primary variables
Dependency		Independent/ and Any Dependency
Linear Mechanics		Non-linear materials, Large strain

Pbox representation for Imprecise Probability



P-Box Finite Element Calculations

Interval Monte Carlo

Direct Calculations (Discrete P-boxes)

P-Box by Interval Monte Carlo

- P box loading
- P box parameters
- Non linear problems
- Independent uncertain variables

P-Box by Direct Methods

- P-box bounday conditions
- P-box parameters
- Non linear problems
- Any Dependency

Copulas

Copulas define dependency between random variable with any marginal distribution.

The marginal distributions are transformed to uniform distribution between [0,1], the Copula is then the CDF of the transformed (ie. Uniform) variables.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. Publications de L'Institut de Statistiques de l'Université de Paris 8: 229-231.

Fréchet-Hoeffding copula bounds

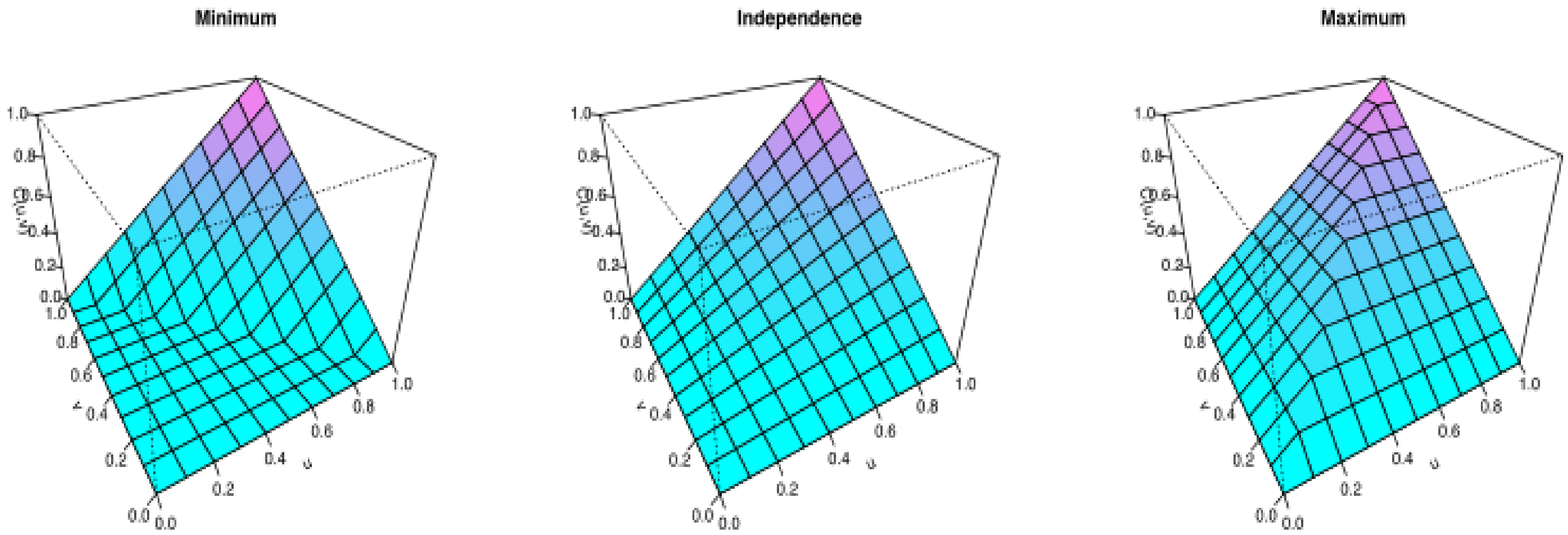


Figure by Matteo Zandi,”Graph of the Fréchet-Hoeffding copula limit
Wikimedia commons

Fréchet-Hoeffding copula bounds

$$W(u_1, \dots, u_d) \leq C(u_1, \dots, u_d) \leq M(u_1, \dots, u_d).$$

$$W(u_1, \dots, u_d) = \max\left\{1 - d + \sum_{i=1}^{d} u_i, 0\right\}.$$

$$M(u_1, \dots, u_d) = \min\{u_1, \dots, u_d\}.$$

Fréchet–Hoeffding Copulas allow for bounding operation on RV with given marginal distributions with no assumption on dependency

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Finite Element Equations

$$\mathrm{f} = Ku$$

$$\mathbf{K}^P = \mathbf{A}\,\mathrm{diag}\left(\boldsymbol{\Lambda}\boldsymbol{\alpha}^P\right)\mathbf{A}^T, \qquad \mathbf{f}^P = \mathbf{F}\boldsymbol{\delta}^P,$$

$$\mathbf{K}_e^P = \begin{Bmatrix} E^P A/L & 0 & -E^P A/L & 0 \\ 0 & 0 & 0 & 0 \\ -E^P A/L & 0 & E^P A/L & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

$$\mathbf{A}_e = \{-1 \;\; 0 \;\; 1 \;\; 0\}^T, \qquad \boldsymbol{\Lambda}_e = \{A/L\}, \qquad \boldsymbol{\alpha}_e^P = \{E^P\}$$

Final System of Linear P-box equations

$$\mathbf{u}^P = \begin{Bmatrix} \mathbf{u}_e^P \\ \vdots \\ \mathbf{u}_e^P \\ \mathbf{u}_n^P \end{Bmatrix}, \qquad \mathbf{K}^P = \begin{Bmatrix} \mathbf{K}_e^P & & & \\ & \ddots & & \\ & & \mathbf{K}_e^P & \\ & & & 0 \end{Bmatrix}, \qquad \mathbf{f}^P = \begin{Bmatrix} \mathbf{f}_e^P \\ \vdots \\ \mathbf{f}_e^P \\ \mathbf{f}_n^P \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbf{K}^P & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \mathbf{u}^P \\ \boldsymbol{\lambda}^P \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^P \\ \mathbf{0} \end{Bmatrix}$$

$$\left(\begin{Bmatrix} \mathbf{A} \\ \mathbf{0} \end{Bmatrix} \operatorname{diag}(\boldsymbol{\Lambda}\boldsymbol{\Delta}\boldsymbol{\alpha}^P) \begin{Bmatrix} \mathbf{A}^T & \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{K}_0 & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{Bmatrix} \right) \begin{Bmatrix} \mathbf{u}^P \\ \boldsymbol{\lambda}^P \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \end{Bmatrix} \delta^P.$$

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Fixed point form

$$\mathbf{K}_{g0}\mathbf{u}_g^P = \mathbf{F}_g\boldsymbol{\delta}^P - \mathbf{A}_g \operatorname{diag}\left(\boldsymbol{\Lambda}\boldsymbol{\Delta}\boldsymbol{\alpha}^P\right)\mathbf{A}_g^T\mathbf{u}_g^P.$$

By defining $\mathbf{G} = \mathbf{K}_{g0}^{-1}$, the above equation is rewritten into the following fixed-point form

$$\mathbf{u}_g^P = (\mathbf{G}\mathbf{F}_g)\boldsymbol{\delta}^P - (\mathbf{G}\mathbf{A}_g)\left(\mathbf{A}_g^T\mathbf{u}_g^P \circ \boldsymbol{\Lambda}\boldsymbol{\Delta}\boldsymbol{\alpha}^P\right),$$

where the following equality has been used

$$\operatorname{diag}\left(\boldsymbol{\Lambda}\boldsymbol{\Delta}\boldsymbol{\alpha}^P\right)\mathbf{A}_g^T\mathbf{u}_g^P = \operatorname{diag}\left(\mathbf{A}_g^T\mathbf{u}_g^P\right)\boldsymbol{\Lambda}\boldsymbol{\Delta}\boldsymbol{\alpha}^P = \mathbf{A}_g^T\mathbf{u}_g^P \circ \boldsymbol{\Lambda}\boldsymbol{\Delta}\boldsymbol{\alpha}^P,$$

Iterative solution

$$\mathbf{v}^P = \mathbf{A}_g^T \mathbf{u}_g^P$$

$$\mathbf{v}^P = (\mathbf{A}_g^T \mathbf{G} \mathbf{F}_g)\delta^P - (\mathbf{A}_g^T \mathbf{G} \mathbf{A}_g)\left(\mathbf{v}^P \circ \mathbf{\Lambda} \mathbf{\Delta} \alpha^P\right)$$

$$\mathbf{v}_0^P = (\mathbf{A}_g^T \mathbf{G} \mathbf{F}_g)\delta^P$$

$$\mathbf{v}_{i+1}^P = (\mathbf{A}_g^T \mathbf{G} \mathbf{F}_g)\delta^P - (\mathbf{A}_g^T \mathbf{G} \mathbf{A}_g)\left(\mathbf{v}_i^P \circ \mathbf{\Lambda} \mathbf{\Delta} \alpha^P\right)$$

$$\mathbf{u}_g^P = (\mathbf{G} \mathbf{F}_g)\delta^P - (\mathbf{G} \mathbf{A}_g)\left(\mathbf{A}_g^T \mathbf{u}_g^P \circ \mathbf{\Lambda} \mathbf{\Delta} \alpha^P\right)$$

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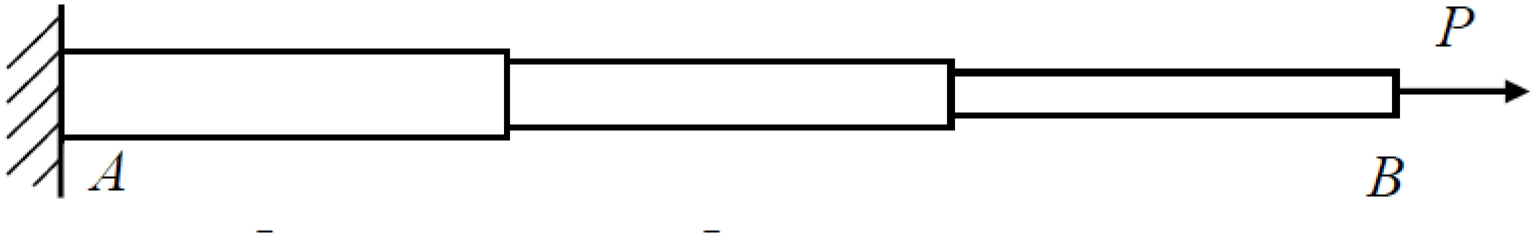
A fixed-end bar subject to axial load at the free end

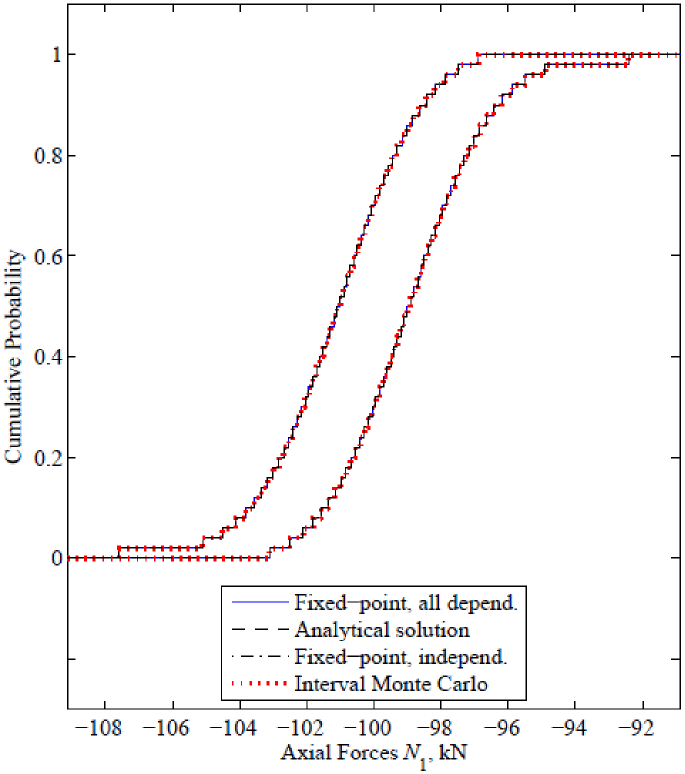


Properties

Table I. Bounds on the mean and standard deviation of the contained CDF's in the p-boxes. P - concentrated load; E_i - Young's moduli of the bar in Figure 1.

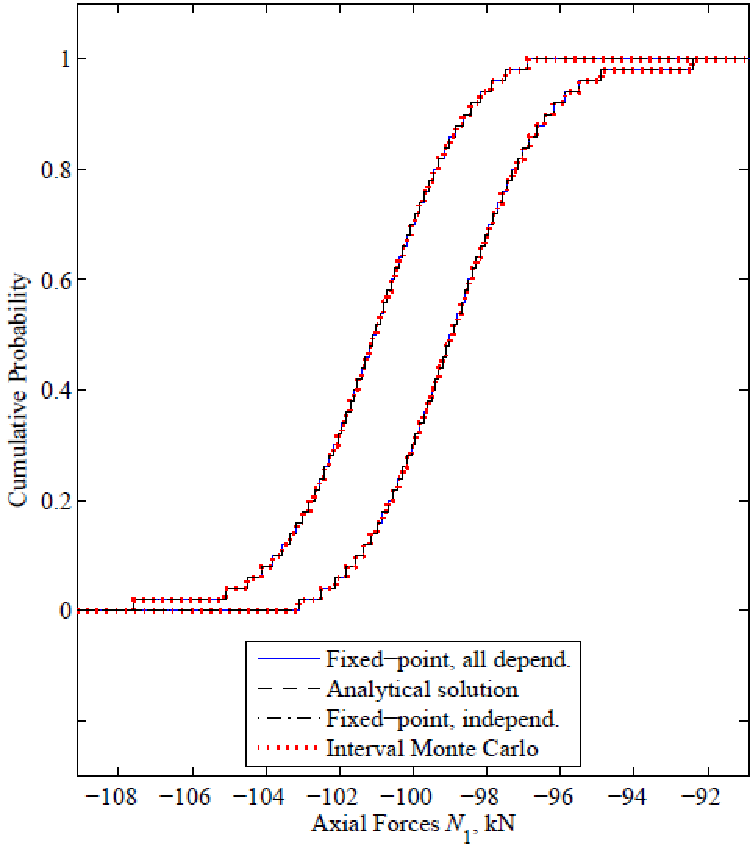
	Concentrated load, kN			Young's moduli, GPa	
	Mean μ_P	Standard deviation σ_P		Mean μ_E	Standard deviation σ_E
P	[98.86, 101.14]	[1.142, 3.136]	E_1	[108.9, 111.1]	[1.142, 3.136]
			E_2	[108.9, 111.1]	[1.142, 3.136]
			E_3	[108.9, 111.1]	[1.142, 3.136]

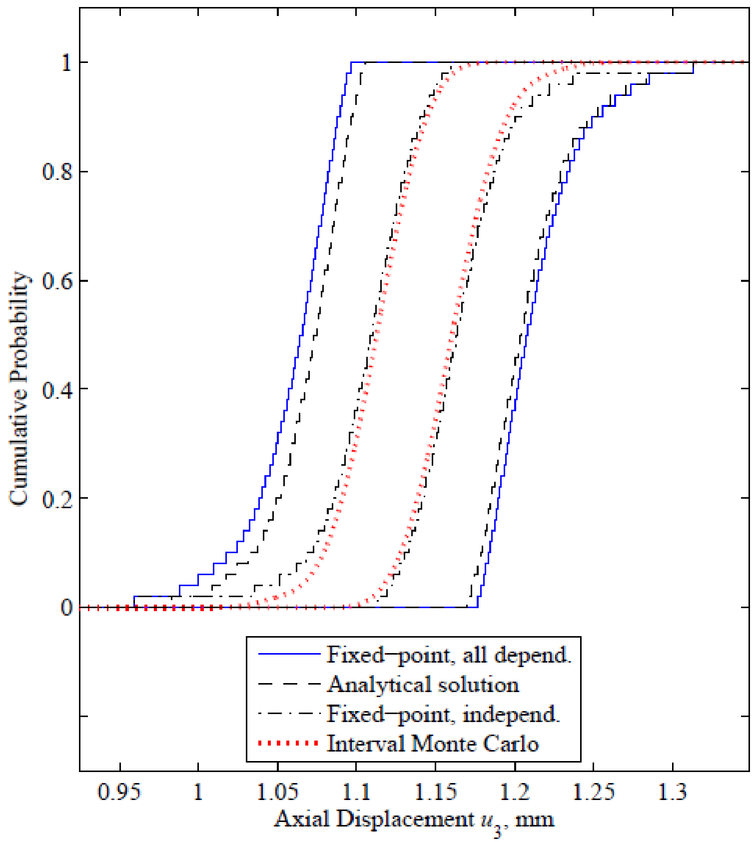
Results



Cumulative Probability
0
0.2
0.4
0.6
0.8
1
−108
−106
−104
−102
−100
−98
−96
−94
−92
Axial Forces N_1, kN
Fixed−point, all depend.
Analytical solution
Fixed−point, independ.
Interval Monte Carlo

Results (2)


Cumulative Probability
1
0.8
0.6
0.4
0.2
0
−108 −106 −104 −102 −100 −98 −96 −94 −92
Axial Forces N_1, kN
Fixed−point, all depend.
Analytical solution
Fixed−point, independ.
Interval Monte Carlo


Cumulative Probability
1
0.8
0.6
0.4
0.2
0
0.95 1 1.05 1.1 1.15 1.2 1.25 1.3
Axial Displacement u_3, mm
Fixed−point, all depend.
Analytical solution
Fixed−point, independ.
Interval Monte Carlo

Results(3)

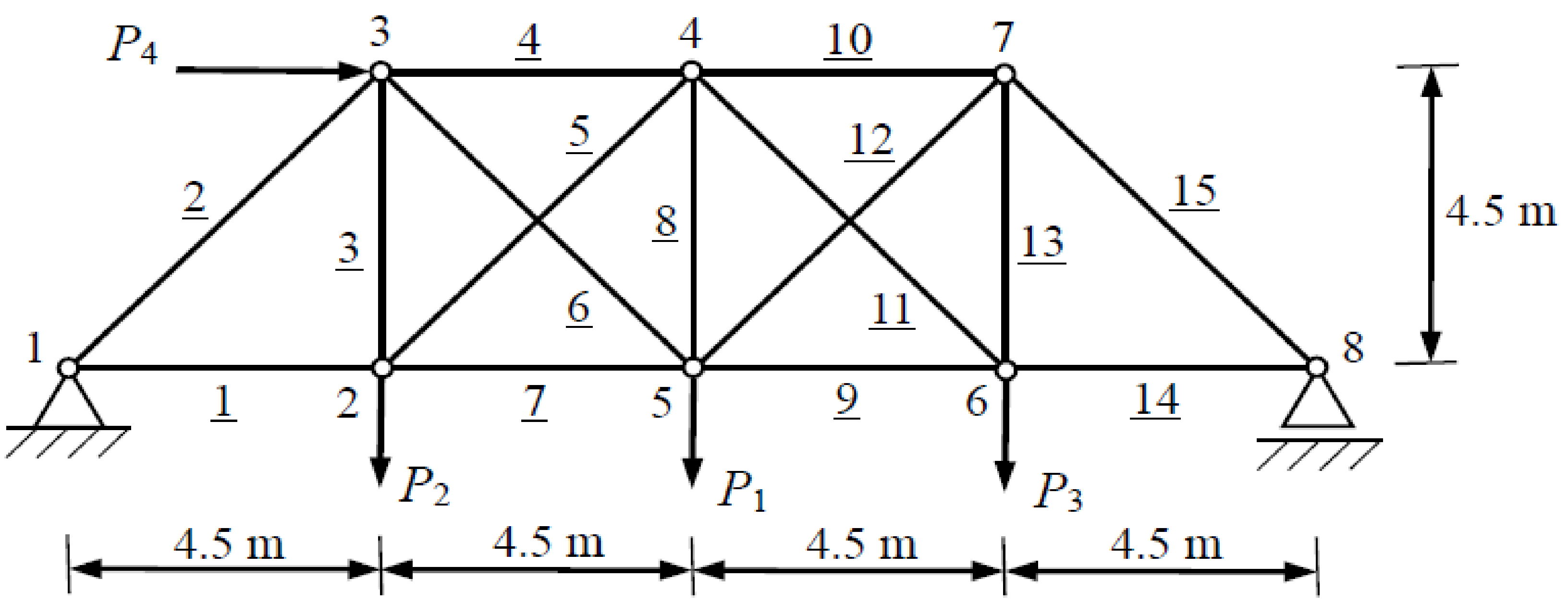
Table II. Bounds on the mean and standard deviation of the contained CDF's in the p-boxes. N_1 - axial force; u_i - axial displacements of the bar in Figure 1.

	Axial force N_1, kN		Axial displacement u_1, mm	
	Mean μ_{N_1}	Standard deviation σ_{N_1}	Mean μ_{u_1}	Standard deviation σ_{u_1}
Current, all depend.	[−101.13, −98.87]	[1.142, 3.136]	[0.2864, 0.3197]	[0.0000, 0.0244]
Reference (analytical)	[−101.13, −98.87]	[1.142, 3.136]	[0.2880, 0.3190]	[0.0000, 0.0227]
Current, independ.	[−101.13, −98.87]	[1.142, 3.136]	[0.2953, 0.3108]	[0.0030, 0.0172]
Interval Monte Carlo	[−101.14, −98.88]	[1.127, 3.096]	[0.2965, 0.3098]	[0.0036, 0.0145]
	Axial displacement u_2, mm		Axial displacement u_3, mm	
	Mean μ_{u_2}	Standard deviation σ_{u_2}	Mean μ_{u_3}	Standard deviation σ_{u_3}
Current, all depend.	[0.6381, 0.7255]	[0.0000, 0.0595]	[1.0586, 1.2141]	[0.0000, 0.1033]
Reference (analytical)	[0.6436, 0.7232]	[0.0000, 0.0551]	[1.0689, 1.2100]	[0.0000, 0.0954]
Current, independ.	[0.6637, 0.6999]	[0.0048, 0.0380]	[1.1055, 1.1673]	[0.0068, 0.0631]
Interval Monte Carlo	[0.6672, 0.6970]	[0.0063, 0.0303]	[1.1120, 1.1617]	[0.0094, 0.0482]

Computer time

	Fixed-end bar (s)
Fixed-point, all dependency	0.41
Fixed-point, independent	0.07
Interval Monte Carlo	573.93
Analytical solution	0.01

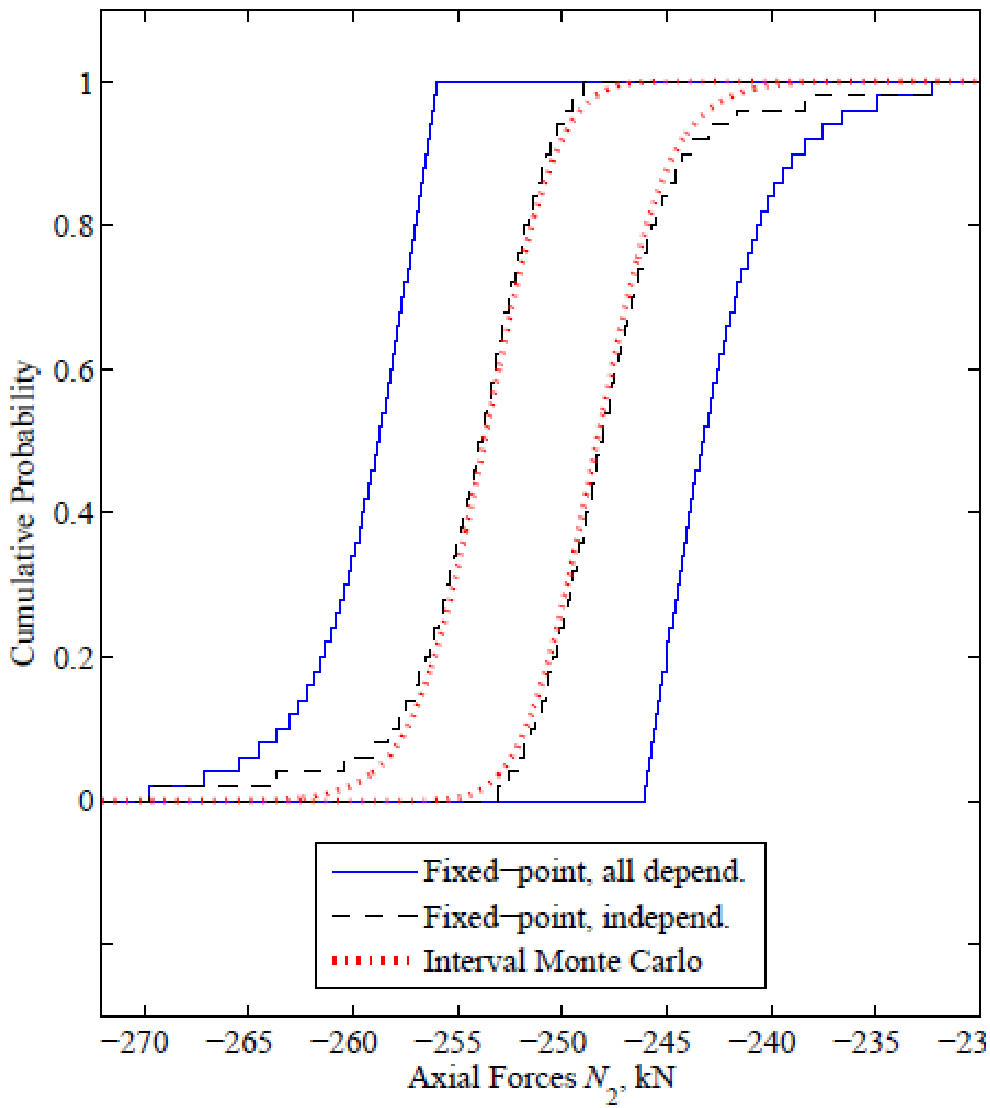
An 8-joint 15-bar symmetric simple truss subject to point loads.



Uncertain Parameters

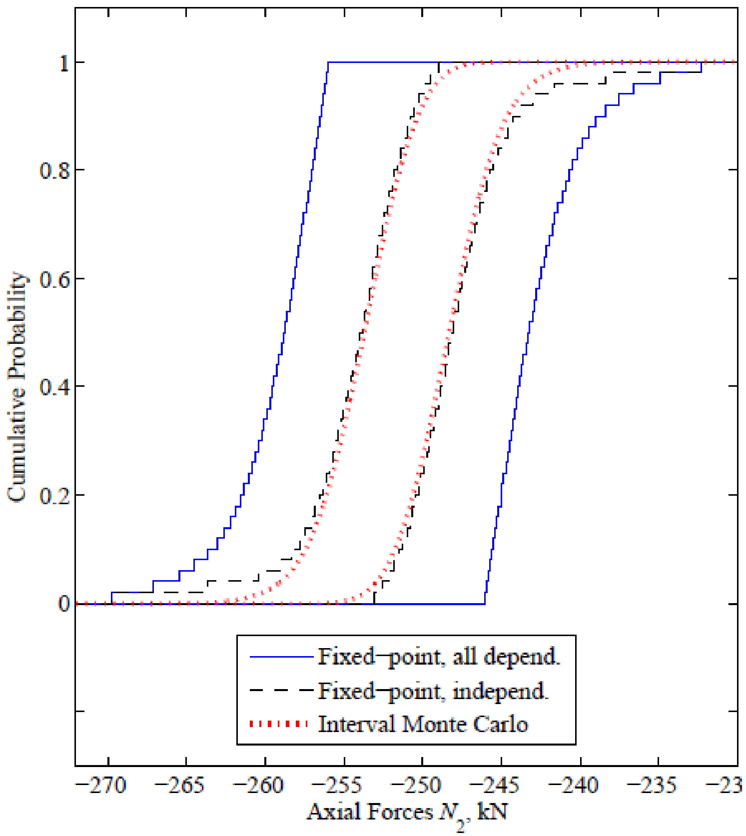
Table IV. Bounds on the mean and standard deviation of the contained CDF's in the p-boxes. P_i - concentrated loads; E_i - Young's moduli of the truss in Figure 3.

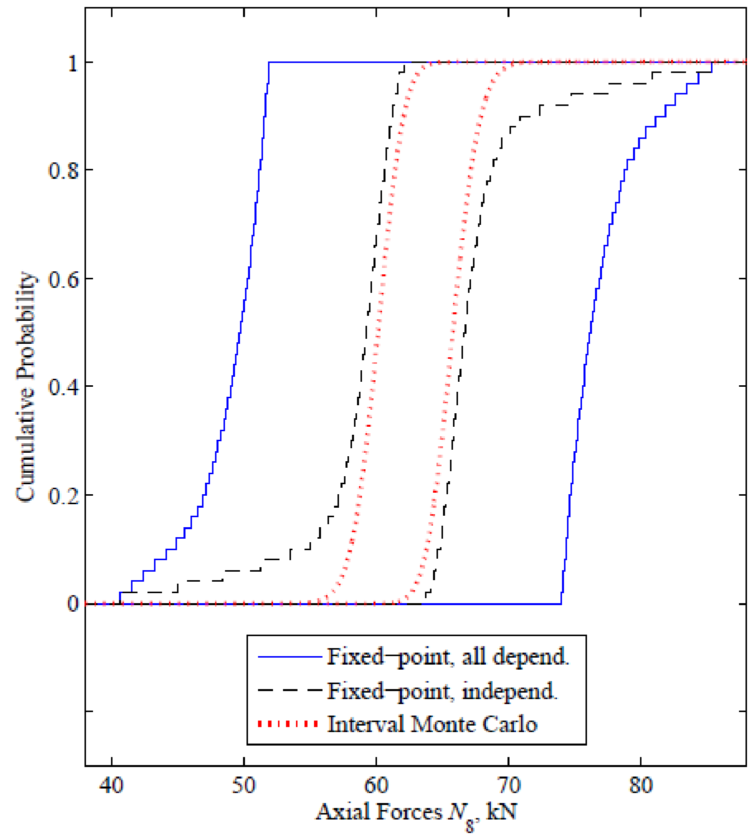
	Concentrated load, kN			Young's moduli, GPa	
	Mean μ_P	Standard deviation σ_P		Mean μ_E	Standard deviation σ_E
P_1	[198.87, 201.13]	[1.142, 3.136]	E_i	[197.7, 202.3]	[2.285, 6.271]
P_2	[98.87, 101.13]	[1.142, 3.136]			
P_3	[98.87, 101.13]	[1.142, 3.136]			
P_4	[88.87, 91.13]	[1.142, 3.136]			



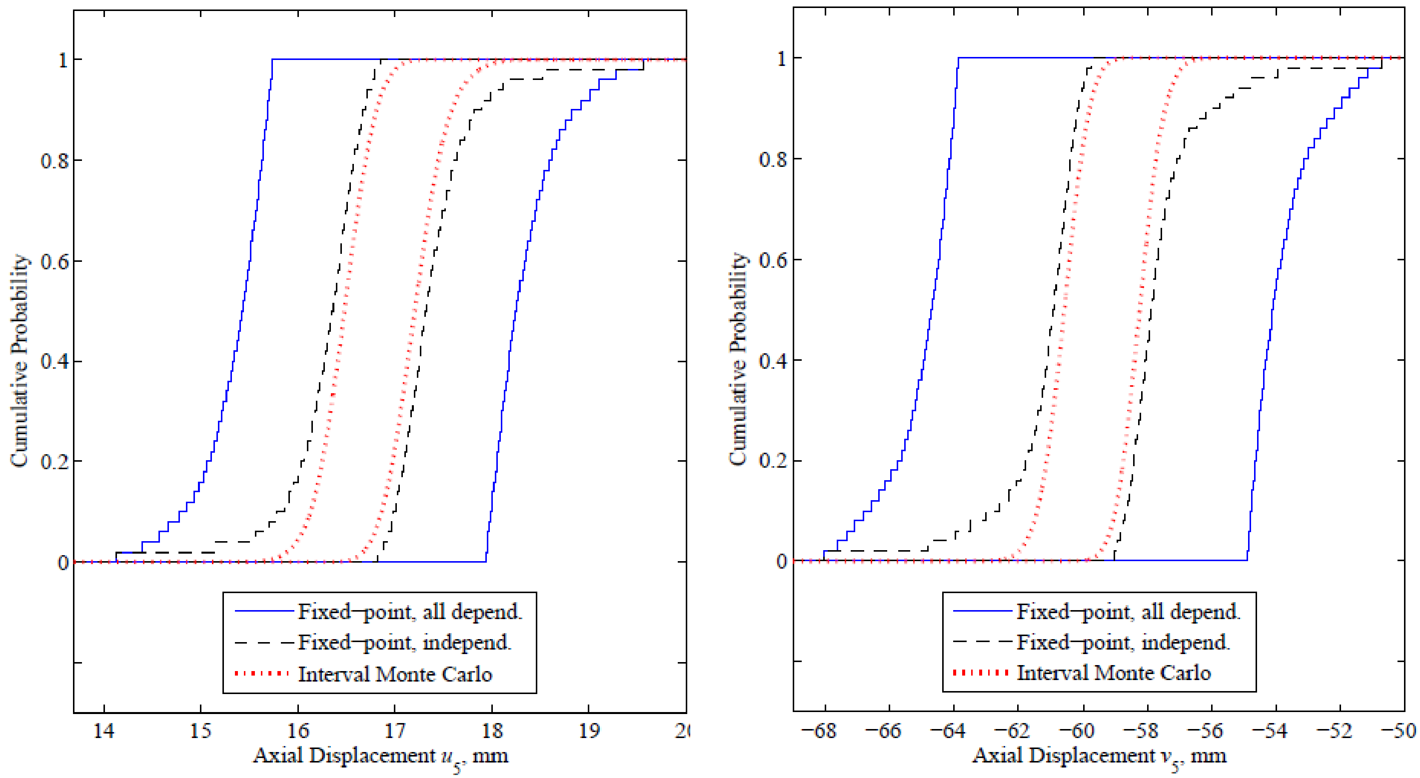
Cumulative Probability
0
0.2
0.4
0.6
0.8
1
−270
−265
−260
−255
−250
−245
−240
−235
−23
Axial Forces N_2, kN
Fixed−point, all depend.
Fixed−point, independ.
Interval Monte Carlo

Element Forces





Displacements



Computer time

Table III. Computational time of the fixed-end bar (Example 4.1) and the simple truss (Example 4.2) for different methods.

	Fixed-end bar (s)	Simple truss (s)
Fixed-point, all dependency	0.41	13.20
Fixed-point, independent	0.07	2.27
Interval Monte Carlo	573.93	907.68
Analytical solution	0.01	N/A

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Conclusions

- A method for solving a system of linear equations with p-box entries in a form consisted with finite element methods is presented
- Numerical results indicate the method provides reasonable bounds on CDF.

Conclusions

- The method allows for dependency bounds to be determined from problem parameters.
- This method may form the foundation for extensions to General nonlinear finite element calculations