Approximation Concepts for Fuzzy Analysis in Structural Dynamics

Marcos Valdebenito[1)], Camilo Pérez[1)], Héctor Jensen[1)] and Michael Beer[2)]
[1)] *Department of Civil Engineering, Santa Maria University, Valparaiso, Chile*
{marcos.valdebenito,hector.jensen}@usm.cl
[2)] *Institute for Risk and Reliability, Leibniz Universität Hannover, Hannover, Germany / University of Liverpool, Liverpool, UK / Tongji University, Shanghai, China*
beer@irz.uni-hannover.de

Abstract: This contribution proposes a framework for performing fuzzy analysis for problems of structural dynamics. In order to avoid repeated structural analyses, a meta-model is used to represent the frequency response function (FRF) approximately. This meta-model comprises two different approximation levels. In the first level, approximate spectral properties (natural frequencies and mode shapes) are considered for calculating the FRF. In the second level, the mode shapes are approximated using a linear expansion with respect to the fuzzy variables of the problem, while the natural frequencies are also represented using a linear expansion but with respect to *intervening variables*. The fuzzy problem is solved using α-level optimization. A numerical example illustrates the accuracy and efficiency of the proposed scheme

Keywords: fuzzy analysis, structural dynamics, approximation concepts, intervening variables

1. Introduction

The so-called non traditional approaches for uncertainty quantification in engineering have gained considerable attention in the last years (Möller et al., 2000). In particular, interval analysis and fuzzy analysis offer the possibility of addressing problems where there is lack of knowledge or imprecision on the input parameters affecting the performance of a system. In this context, fuzzy analysis can be understood as a sequence of interval analyses. The value of applying fuzzy analysis is that it allows to identify sensitivities of the response of a structure with respect to the magnitude of imprecision of the input.

In spite of the evident advantages of fuzzy structural analysis, its application is not widespread. This is due to the fact that fuzzy analysis demands significant numerical efforts. In such scenario, this contribution presents an approach for performing fuzzy analysis of a particular class of problems, which is most efficient. The class of problems considered involve linear dynamical structural systems, which are analyzed using the frequency response function (FRF).

The efficient solution of problems of fuzzy structural dynamics has been the object of active research in the last few years. The approaches proposed for solving this problem can be broadly classified into two groups. The first group includes approaches that apply interval arithmetic. For example, in Modares et al., 2006, the intervals associated with natural frequencies of a structure comprising interval parameters are calculated exactly using only two structural analyses by taking advantage of a formulation based on the Rayleigh quotient. Dessombz et al., 2001 apply an iterative algorithm for assessing the intervals associated with FRF functions; the algorithm addresses dependency issues of interval arithmetic. Manson, 2005 applies complex affine analysis for estimating the fuzzy FRF. Sim et al., 2007 propose an approach for calculating the FRF by considering exact bounds for the natural frequencies plus a first order Taylor expansion of the mode shapes. Muscolino et al., 2014 consider a *rational series expansion* in order to produce an accurate approximation of the FRF as an explicit function of the fuzzy variables.

The second group of strategies for fuzzy dynamic analysis includes approaches that apply optimization techniques, which attempt to identify the extrema of the structural response. For example, in Adhikari et al., 2011, the application of high dimensional model representation has been investigated in order to generate an explicit approximation of the fuzzy responses of interest. In Beer and Liebscher, 2008, a general framework for design under fuzziness is presented, which combines cluster analysis, an appropriate optimization strategy and robustness assessment. Adhikari and Khodaparast, 2014 propose a framework for fuzzy analysis combining the so-called Polynomial Chaos Expansion (PCE) in combination with a mode-based reduction strategy. Massa et al., 2008 propose an approach for solving fuzzy structural dynamics problems where natural frequencies and mode shapes are approximated using Padé rational functions. In addition to the two groups of strategies for fuzzy dynamic analysis mentioned above, it is interesting to note that a hybrid approach has been proposed by De Munck et al., 2008, where modal interval analysis (Moens and Vandepitte, 2004) and a response surface are applied simultaneously for estimating the intervals associated with the FRF.

The approach proposed in this contribution belongs to the second group mentioned above, i.e. it is formulated within the context of the α-level optimization strategy (Möller et al., 2000). In order to identify the minimum and maximum values of the structural response (for a given membership value α), the optimization algorithm introduced in (Li and Au, 2010) is considered, which is based on Subset Simulation (Au and Beck, 2001). In addition, the structural response is calculated approximately using a meta-model which approximates the frequency response function (FRF) explicitly with respect to the fuzzy input variables. This meta-model comprises two different approximation levels. In the first level, approximate spectral properties (natural frequencies and mode shapes) are considered for calculating the FRF. In the second level, the mode shapes are approximated using a linear expansion with respect to the fuzzy variables of the problem, while the natural frequencies are also represented using a linear expansion but with respect to *intervening variables* (Schmit and Farshi, 1974). The application of the latter type of variables has already been shown to provide accurate approximations within the context of fuzzy analysis for static structures (Valdebenito et al., 2016).

2. Formulation of the Problem

2.1. Fuzzy Variables and Structural Model

Consider a linear elastic structure modeled using the FE method (Bathe, 1996). In order to characterize this structure, there are $x_i,\ i = 1, \ldots, N_p$ fuzzy variables associated with structural properties (e.g. Young's modulus, cross section area, etc.) and $y_j,\ j = 1, \ldots, N_l$ fuzzy parameters associated with loads acting over the structure; $\mu_{\tilde{x}_i}(x_i),\ i = 1, \ldots, N_p$ and $\mu_{\tilde{y}_j}(y_j),\ j = 1, \ldots, N_l$ represent the membership functions associated with these fuzzy variables. It is assumed that the membership functions are convex. In the following, the fuzzy variables associated with structural properties and loads are grouped in vectors $\boldsymbol{x} = \langle x_1, x_2, \ldots, x_{N_p}\rangle^T$ and $\boldsymbol{y} = \langle y_1, y_2, \ldots, y_{N_l}\rangle^T$, respectively.

The structural system under study is characterized considering a total of N_d degrees-of-freedom. Then, the equation of motion relating structural properties, loads, displacements, velocities and accelerations of the FE model of the structure is:

$$\boldsymbol{M}(\boldsymbol{x})\ddot{\boldsymbol{u}}(t, \boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{C}(\boldsymbol{x})\dot{\boldsymbol{u}}(t, \boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}(t, \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{f}(t, \boldsymbol{y}) \tag{1}$$

where t represents time; $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{C}(\boldsymbol{x})$ and $\boldsymbol{K}(\boldsymbol{x})$ are the mass, damping and stiffness matrices (dimension $N_d \times N_d$), where it is assumed $\boldsymbol{C}(\boldsymbol{x})$ corresponds to a classical damping matrix; $\ddot{\boldsymbol{u}}(t, \boldsymbol{x}, \boldsymbol{y})$, $\dot{\boldsymbol{u}}(t, \boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{u}(t, \boldsymbol{x}, \boldsymbol{y})$ are the acceleration, velocity and displacement vector (dimension $N_d \times 1$); and $\boldsymbol{f}(t, \boldsymbol{y})$ is the load vector (dimension $N_d \times 1$). As both structural matrices and load vector depend on fuzzy variables, the structural response (e.g. displacement) also depends on these variables, i.e. the fuzziness in the structural properties and load propagates to the structural response. However, it should be noted that the membership function associated with the structural response cannot be calculated analytically (for almost all cases of practical interest).

Assume the load vector $\boldsymbol{f}(t, \boldsymbol{y})$ models a periodic loading and that it is of interest to determine the membership function associated with the steady-state displacement vector $\boldsymbol{u}_P(t, \boldsymbol{x}, \boldsymbol{y})$. Such problem can be solved most conveniently using the frequency response function (FRF), as described in the following. As the load vector is periodic, then it can be described as:

$$\boldsymbol{f}(t, \boldsymbol{y}) = \sum_{l=1}^{N_f} \boldsymbol{f}_l(\boldsymbol{y}) e^{i\omega_l(\boldsymbol{y})t} \tag{2}$$

where $\boldsymbol{f}_l(\boldsymbol{y})$ denotes the amplitude of the load vector with associated frequency $\omega_l(\boldsymbol{y})$; note both the amplitude and frequency may be dependent on the fuzzy vector $\boldsymbol{y}$. Then, the steady-state displacement vector is equal to:

$$\boldsymbol{u}_P(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{l=1}^{N_f} \boldsymbol{H}(\omega_l(\boldsymbol{y}), \boldsymbol{x}) \boldsymbol{f}_l(\boldsymbol{y}) e^{i\omega_l(\boldsymbol{y})t} \tag{3}$$

where $\boldsymbol{H}(\omega, \boldsymbol{x})$ is the so-called frequency response function (FRF) of dimension $N_d \times N_d$, which is defined as shown below (Craig and Kurdila, 2006).

$$\boldsymbol{H}(\omega, \boldsymbol{x}) = \sum_{r=1}^{N_m} \frac{\boldsymbol{\phi}_r(\boldsymbol{x})\boldsymbol{\phi}_r^T(\boldsymbol{x})}{\omega_r(\boldsymbol{x})^2 - \omega^2 + i2\xi_r\omega\omega_r(\boldsymbol{x})} \tag{4}$$

In the above equation, N_m denotes the number of modes retained for dynamic analysis ($N_m \leq N_d$), ξ_r denotes the damping ratio associated with the r-th mode, while $\omega_r(\boldsymbol{x})$ and $\boldsymbol{\phi}_r(\boldsymbol{x})$ denote the r-th natural frequency and r-th mode shape, respectively. The natural frequencies and mode shapes are calculated by performing a dynamic structural analysis, which in this case involves solving the following eigenvalue/eigenvector problem.

$$\left(\boldsymbol{K}(\boldsymbol{x}) - \omega_r^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x})\right)\boldsymbol{\phi}_r(\boldsymbol{x}) = \boldsymbol{0},\ r = 1, \ldots, N_m \tag{5}$$

It is assumed that the mode shapes are orthonormal with respect to the mass matrix. The solution of the above problem can be numerically demanding, particularly for structures which comprise a large number of degrees-of-freedom.

2.2. α-Level Optimization

Recall the objective pursued is determining the membership function associated with the steady-state displacement vector. A possible means for determining such membership function is applying the so-called α-level optimization. In this manner, the membership function is represented in a discrete way. This implies that the values the displacement vector may assume are calculated at specific α-cuts, where α denotes the membership level under analysis (Beer, 2004); clearly, $0 < \alpha \leq 1$.

For applying α-level optimization, assume a specific cut α_k is selected and that the objective is determining the membership function of the n-th DOF of displacement vector $u_{P,n}$. This implies that the variables associated with structural parameters and loads are contained within the intervals $\underline{x}_{i,\alpha_k}$, $i = 1, \ldots, N_p$ and $\underline{y}_{j,\alpha_k}$, $j = 1, \ldots, N_l$, respectively. Note that under the assumption that the sets $\underline{x}_{i,\alpha_k}$ and $\underline{y}_{j,\alpha_k}$ are compact and convex, these sets are fully described by their minimum and maximum values (denoted with superscripts $(\cdot)^L$ and $(\cdot)^R$, respectively, see Figure 1). As there is a continuous mapping between the variables $(\boldsymbol{x}, \boldsymbol{y})$ and the output variable vector $\boldsymbol{u}_p$ (see Eq.(3)), the set $\underline{u}_{P,n,\alpha_k}$ is also fully described by its minimum and maximum value (denoted with superscripts $(\cdot)^L$ and $(\cdot)^R$, respectively, see Figure 1). These two extrema actually constitute two points of the membership function $\mu_{\tilde{u}_{P,n}}(u_{P,n})$ for the membership level α_k. Note that for determining the set $\underline{u}_{P,n,\alpha_k}$, it is necessary to solve two optimization problems (Moens and Vandepitte, 2005), i.e. minimization and maximization of the displacement given that the variables associated with structural properties and loading belong to the interval defined by their respective α-cuts. In order to visualize the α-level optimization strategy, Figure 1 contains a schematic representation where it is assumed that $N_p = N_l = 1$.

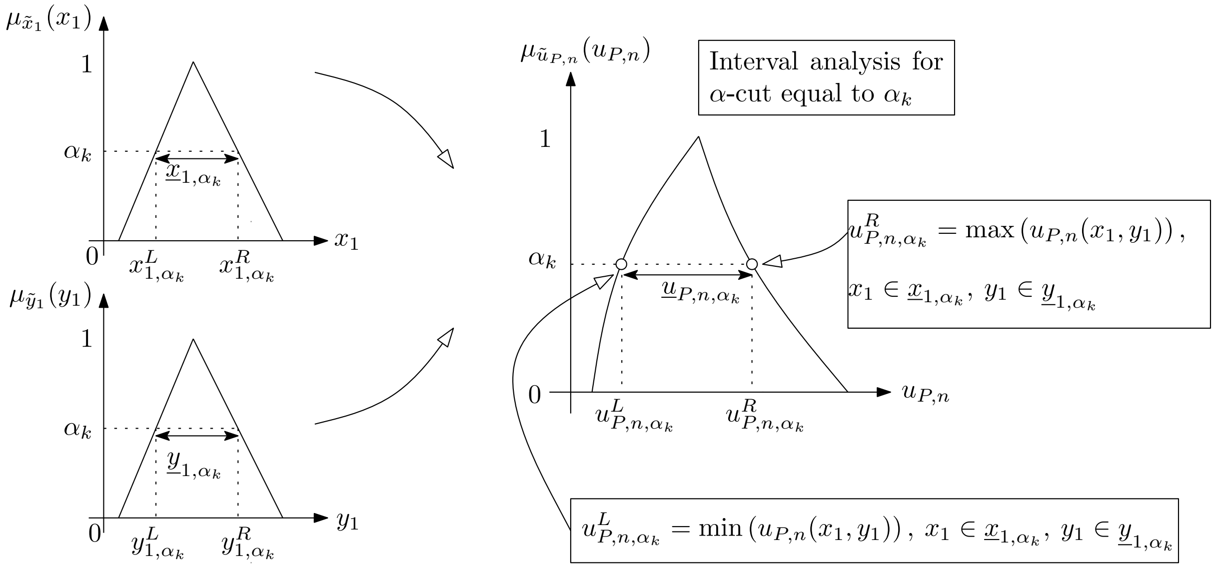

Figure 1. Schematic Representation of Fuzzy Analysis Applying α-level Optimization.

As noted from the above discussion, the application of α-level optimization demands solving two optimization problems (minimization and maximization) for each discrete α-cut being analyzed. For problems of practical interest, the numerical costs associated with this procedure can be considerable and even prohibitive, as it demands performing structural analyses for different values of the fuzzy variables. In order to reduce the aforementioned numerical costs, this contribution introduces a meta-model for the FRF function such that it can be evaluated with negligible effort. The details of the meta-model are discussed in the next section.

3. Approximate Representation of the Frequency Response Function

3.1. First Approximation Level

The most demanding step for evaluating the FRF function is calculating the spectral properties (i.e. natural frequencies and mode shapes). Therefore, in order to decrease the associated numerical costs, it is proposed to use a meta-model for evaluating the FRF that includes approximations of the spectral properties, as shown below.

$$\tilde{\boldsymbol{H}}(\omega, \boldsymbol{x}) = \sum_{r=1}^{N_m} \frac{\tilde{\boldsymbol{\phi}}_r(\boldsymbol{x})\tilde{\boldsymbol{\phi}}_r^T(\boldsymbol{x})}{\tilde{\omega}_r(\boldsymbol{x})^2 - \omega^2 + i2\xi_r\omega\tilde{\omega}_r(\boldsymbol{x})} \tag{6}$$

In the above equation, $\tilde{\boldsymbol{H}}$ denotes the approximate FRF, which is calculated using approximate spectral properties $\tilde{\omega}_r$ and $\tilde{\boldsymbol{\phi}}_r,\ r = 1, \ldots, N_m$.
The approximation proposed in Eq. (6) constitutes actually the first level of approximation of the proposed strategy. This corresponds to approximating an *intermediate quantity* (see, e.g. Jensen,

2000). This is an idea that has been used customarily in the field of structural optimization. Its basis is the following: as the FRF is a highly nonlinear function, attempting to approximate it explicitly in terms of the vector of structural parameters $\boldsymbol{x}$ would result quite challenging. In this context, *challenging* implies that probably higher order terms would be required for producing a sufficiently accurate approximation. In view of this issue, approximating an *intermediate quantity* is much more convenient, as this quantity may behave more linearly with respect to $\boldsymbol{x}$. In the context of the proposed meta-model, the intermediate quantity is actually the set of spectral properties.

3.2. Second Approximation Level

For the actual implementation of the meta-model proposed in Eq. (6), it is necessary to compute the approximate spectral properties. For approximating the mode shapes, it is proposed to apply a first order Taylor expansion, i.e.:

$$\tilde{\boldsymbol{\phi}}_r(\boldsymbol{x}) = \boldsymbol{\phi}_r\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_p} \left.\frac{\partial \boldsymbol{\phi}_r}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^0} \left(x_i - x_i^0\right) \tag{7}$$

where $\boldsymbol{x}^0$ is an expansion point.

For approximating the natural frequencies, a first order Taylor expansion is considered as well. However, instead of formulating the expansion directly with respect to $\boldsymbol{x}$, it is cast with respect to *intervening variables* $I_{r,i}(x_i)$, $i = 1, \ldots, N_p$ (Prasad, 1983). These intervening variables are nonlinear functions with respect to x_i. The advantage of considering this strategy is that the quantity being approximated (in this case, the natural frequencies) behave more linearly with respect to $I_{r,i}(x_i)$ than with respect to x_i. Therefore, the approximation for the natural frequencies is the following.

$$\tilde{\omega}_r(\boldsymbol{x}) = \omega_r\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_p} \left.\frac{\partial \omega_r}{\partial I_{r,i}}\right|_{\boldsymbol{x}=\boldsymbol{x}^0} \left(I_{r,i}\left(x_i\right) - I_{r,i}\left(x_i^0\right)\right) \tag{8}$$

For the case of fuzzy static structural analysis, it has been showed the so-called *exponential* intervening variable is most useful (Valdebenito et al., 2016). The exponential intervening variable possesses the form:

$$I_{r,i}\left(x_i\right) = x_i^{c_{r,i}} \tag{9}$$

where $c_{r,i}$ is a real constant. Thus, the approximation for the natural frequencies reduces to the following expression.

$$\tilde{\omega}_r(\boldsymbol{x}) = \omega_r\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{N_p} \left.\frac{\partial \omega_r}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^0} \frac{\left(x_i^0\right)^{1-c_{r,i}}}{c_{r,i}} \left(\left(x_i\right)^{c_{r,i}} - \left(x_i^0\right)^{c_{r,i}}\right) \tag{10}$$

In order to select the value of the constant $c_{r,i}$, the following criterion is proposed: the second-order derivatives of the exact natural frequency ω_r and of the approximate frequency $\tilde{\omega}_r$ should be equal at the expansion point. In mathematical terms, this criterion reads as follows.

$$\left.\frac{\partial^2 \omega_r}{\partial x_i^2}\right|_{\boldsymbol{x}=\boldsymbol{x}^0} = \left.\frac{\partial^2 \tilde{\omega}_r}{\partial x_i^2}\right|_{\boldsymbol{x}=\boldsymbol{x}^0}, \ i = 1, \ldots, N_p \tag{11}$$

It can be shown that this criterion leads to the following expression for determining $c_{r,i},\ i = 1,\ldots,N_p$.

$$c_{r,i} = 1 + x_i^0 \frac{\left.\frac{\partial^2 \omega_r}{\partial x_i^2}\right|_{\boldsymbol{x}=\boldsymbol{x}^0}}{\left.\frac{\partial \omega_r}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^0}},\ i = 1,\ldots,N_p \tag{12}$$

As the above equation may render values of $c_{r,i}$ which are too high in case the first derivative of the natural frequency is close to zero, it is suggested to limit the values $c_{r,i}$ may adopt within the range $[-3, 3]$.

3.3. Sensitivity of Natural Frequencies and Mode Shapes

The practical implementation of the approximations described in Section 3.2 demands evaluating the sensitivity of the spectral properties with respect to $\boldsymbol{x}$. The latter is a problem which is the subject of active research, see e.g. Lin and Lim, 1993. In particular, in this contribution, the approach proposed by Nelson, 1976 is implemented. A salient feature of this approach is that for calculating the sensitivity of the r-th natural frequency and mode shape, it requires only the information related with that frequency and mode shape. This is most convenient from a numerical viewpoint, as for large structural systems, usually $N_m \ll N_d$.

For presenting Nelson's method, consider Eq. (5). After differentiating this equation with respect to x_i and pre-multiplication by $\boldsymbol{\phi}_r^T(\boldsymbol{x})$, it is possible to determine the following explicit expression for the sensitivity of the natural frequency.

$$\frac{\partial \omega_r(\boldsymbol{x})}{\partial x_i} = \frac{1}{2\omega_r(\boldsymbol{x})} \boldsymbol{\phi}_r^T(\boldsymbol{x}) \left(\frac{\partial \boldsymbol{K}(\boldsymbol{x})}{\partial x_i} - \omega_r^2(\boldsymbol{x}) \frac{\partial \boldsymbol{M}(\boldsymbol{x})}{\partial x_i} \right) \boldsymbol{\phi}_r(\boldsymbol{x}) \tag{13}$$

In order to determine the sensitivity of the mode shape, consider again consider Eq. (5). After differentiating this equation with respect to x_i and arranging terms, it is possible to find the following expression.

$$\big(\boldsymbol{K}(\boldsymbol{x}) - \omega_r^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x})\big) \frac{\partial \boldsymbol{\phi}_r(\boldsymbol{x})}{\partial x_i} = -\left(\frac{\partial \boldsymbol{K}(\boldsymbol{x})}{\partial x_i} - \omega_r^2(\boldsymbol{x}) \frac{\partial \boldsymbol{M}(\boldsymbol{x})}{\partial x_i} \right) \boldsymbol{\phi}_r(\boldsymbol{x}) + 2\omega_r(\boldsymbol{x}) \frac{\partial \omega_r(\boldsymbol{x})}{\partial x_i} \boldsymbol{M}(\boldsymbol{x}) \boldsymbol{\phi}_r(\boldsymbol{x}) \tag{14}$$

For the above equation, assume that matrix $\big(\boldsymbol{K}(\boldsymbol{x}) - \omega_r^2(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x})\big)$ is denoted as $\boldsymbol{G}_r$ and that all terms of the right hand side are grouped in a vector $\boldsymbol{g}_{r,i}$. Then, the equation can be rewritten as follows.

$$\boldsymbol{G}_r \frac{\partial \boldsymbol{\phi}_r(\boldsymbol{x})}{\partial x_i} = \boldsymbol{g}_{r,i} \tag{15}$$

Note that $\boldsymbol{G}_r$ possesses rank $N_d - 1$ and nullspace $\boldsymbol{\phi}(\boldsymbol{x})$. Therefore, the sought sensitivity of the mode shape can be expressed as:

$$\frac{\partial \boldsymbol{\phi}_r(\boldsymbol{x})}{\partial x_i} = \boldsymbol{h}_{r,i} + \gamma_{r,i} \boldsymbol{\phi}_r(\boldsymbol{x}) \tag{16}$$

where $\boldsymbol{h}_{r,i}$ is any vector which fulfills $\boldsymbol{G}_r \boldsymbol{h}_{r,i} = \boldsymbol{g}_{r,i}$ and $\gamma_{r,i}$ is any real constant. In order to determine $\boldsymbol{h}_{r,i}$ and $\gamma_{r,i}$, the following procedure is proposed by Nelson, 1976.

1. $\boldsymbol{h}_{r,i}$ is determined by setting one of its component equal to some arbitrary value (e.g. equal to 1) and the other components are determined by solving the $N_d - 1$ equations associated with $\boldsymbol{G}_r \boldsymbol{h}_{r,i} = \boldsymbol{g}_{r,i}$.

2. $\gamma_{r,i}$ is determined by differentiating with respect to x_i the equation that ensures orthonormality of the mode shapes with respect to the mass matrix. This criterion yields the following expression for $\gamma_{r,i}$.
$$\gamma_{r,i} = -\boldsymbol{\phi}_r^T(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x})\boldsymbol{h}_{r,i} - \frac{1}{2}\boldsymbol{\phi}_r^T(\boldsymbol{x})\frac{\partial \boldsymbol{M}(\boldsymbol{x})}{\partial x_i}\boldsymbol{\phi}_r(\boldsymbol{x}) \tag{17}$$

It should be noted that the procedure for determining the sensitivity of the spectral properties proposed by Nelson, 1976 is valid whenever the natural frequencies are distinct. In case one or more natural frequencies possess multiplicity larger than one, it is possible to extend Nelson's method, as discussed by Dailey, 1989. The way this extension operates is as follows: the sensitivities associated with natural frequencies with multiplicity larger than 1 are calculated using eqs. (13) and (15), but in a matrix form, i.e. instead of calculating one sensitivity, m sensitivities are calculated simultaneously, where m is the multiplicity of the repeated natural frequency. It should be noted that the extension of Nelson's method proposed by Dailey, 1989 produces the second order derivatives of the natural frequencies as a byproduct. These second order derivatives are used to calculate the constants $c_{r,i}$ associated with the intervening variables (see Eq. (12)).
The interested readership is referred to Nelson, 1976 and Dailey, 1989 for details on the procedure for calculating the sensitivity of the spectral properties.

4. Subset Simulation for Optimization

The practical implementation of the α-level optimization approach demands applying an appropriate optimization algorithm for determining the extrema of the displacement function at each α-cut. The solution of this optimization problem is most challenging, as it may present local optima. In view of this challenge, this contribution applies Subset Simulation for optimization (Li and Au, 2010). This is a gradient free optimization algorithm that applies stochastic search for identifying the optimal solution.
Subset Simulation for optimization is applied following 5 steps. These steps are described briefly in the following, assuming it is applied for solving a maximization problem. For a detailed description, the reader is referred to Li and Au, 2010.

1. Introduce an *instrumental* probability density function (pdf) associated with the variables of the problem (in this case, the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$). It is suggested that this instrumental pdf is set as a truncated Gaussian distribution.

2. Generate N samples of the variables $(\boldsymbol{x}, \boldsymbol{y})$ and evaluate for each of these samples the displacement of the structural system applying the meta-model for the FRF.

3. Select the pN samples of $(\boldsymbol{x}, \boldsymbol{y})$ which possess the largest associated displacement value. Note $0 < p < 1$. The selected samples are termed as *seed* samples.

4. Using the *seed* samples from the previous step and the modified Metropolis algorithm (see Au and Beck, 2001), generate $(1-p)N$ additional samples of $(\boldsymbol{x}, \boldsymbol{y})$ whose associated displacement value is equal or larger than any of the values selected in the previous step.

5. Check whether or not the algorithm has converged to the optimum solution. This is performed by assessing the standard deviation of the N samples of $(\boldsymbol{x}, \boldsymbol{y})$. In case convergence has been achieved, select the sample with largest value of the displacement as the optimal solution. Otherwise, return to step 3.

Figure 2 illustrates schematically the application of Subset Simulation for optimization. For simplicity, it is considered that $N_p = N_l = 1$. As noted from the Figure, first $N = 10$ samples of (x_1, y_1) are generated, which are marked with white circles. From these samples, the 2 samples with largest values of displacement are selected as seeds ($p = 0.2$). Then, using these 2 samples, 8 additional samples are generated by means of the Metropolis algorithm (gray circles).

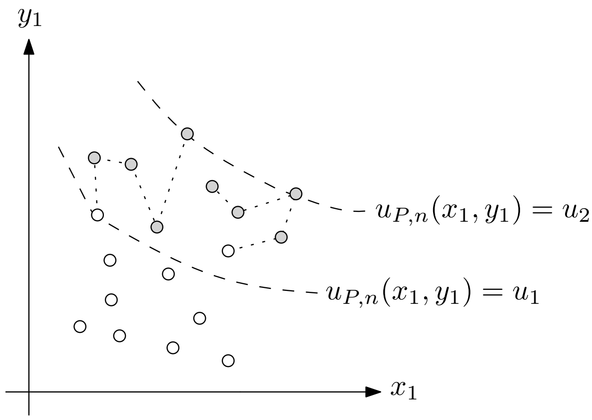

Figure 2. Schematic Representation of Subset Simulation for Optimization.

5. Example

In order to illustrate the application of the proposed strategy for fuzzy structural dynamic analysis, the following example is analyzed, which is taken from Beer and Liebscher, 2008. It consists of a steel beam which possesses distributed mass m and also a concentrated mass M, which is located at a distance l from the left support. The beam is subjected to an harmonic load $f(t, \boldsymbol{y})$, as illustrated in Figure 3. The objective of the example is determining the membership function associated with the maximum amplitude of the steady-state vertical displacement of the beam at the point where the load is applied.

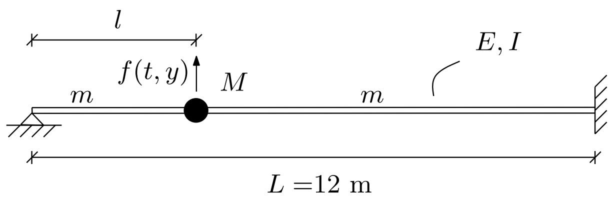

Figure 3. Steel beam subject to point load.

The fuzzy variables associated with the structural parameters of the problem are the position of the concentrated mass $l = x_1$ and the total mass $M_T = x_2$. Note the total mass is the summation of the distributed mass and the concentrated mass, which are defined as $m = M_T/3$ and $M = 2M_T/3$. The membership functions associated with x_1 and x_2 are shown in Figure 4.
The load is described as the superposition of two harmonic signals, i.e.:

$$f(t, \boldsymbol{y}) = y_1 \cos(y_2\omega_1 t) + y_1 \cos(y_2\omega_2 t) \tag{18}$$

where the frequencies are defined as $\omega_1 = 44$ [rad/s] and $\omega_2 = 66$ [rad/s]; and where y_1 and y_2 are fuzzy variables modeling the lack of knowledge on the amplitude and frequency of the loading, respectively. Their membership functions are defined in Figure 4.

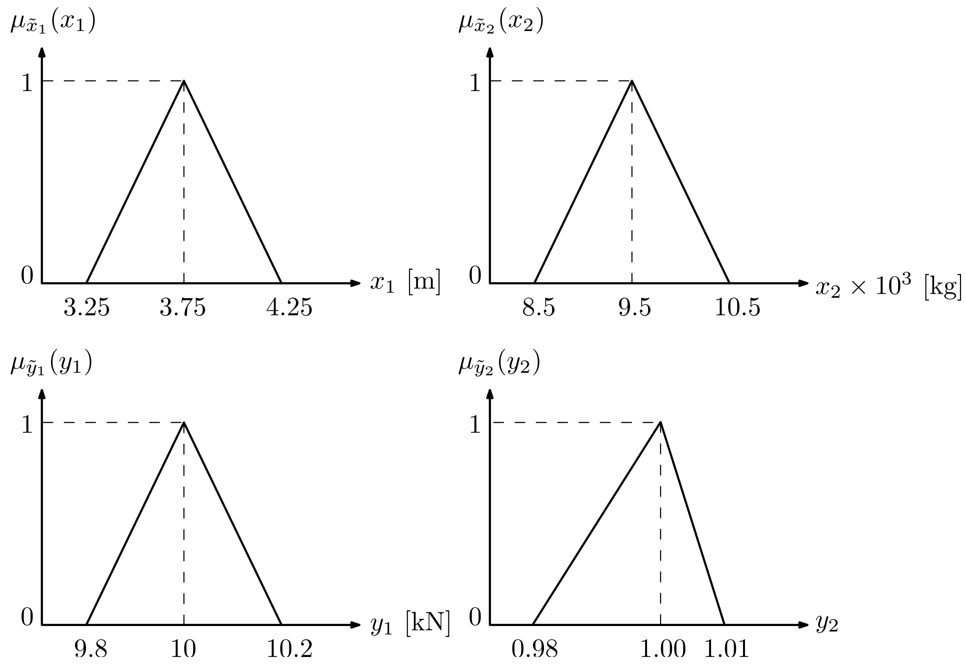

Figure 4. Membership functions of fuzzy variables.

The Young's modulus of the beam is $E = 210$ [GPa] and its second moment of area is $I = 1.5 \times 10^{-3}$ [m^4]. It is assumed that damping is negligible. In order to solve the problem, the beam model is discretized using 12 2D beam elements, which comprise a total of $N_d = 23$ degrees-of-freedom.

Figure 5 shows the results obtained for the problem in terms of the membership function of the maximum amplitude of the steady-state vertical displacement of the beam at the point where the load is applied. The membership function is calculated using the α-level optimization strategy for a total of 11 α-cuts. Subset Simulation for optimization was used to determine the minimum and maximum values of the displacement for each α-cut. Three different strategies were considered for calculating the steady-state displacement.

1. In order to obtain reference results, the displacement was calculated exactly, i.e. a structural analysis was carried out for each different value of the structural parameter vector $\boldsymbol{x}$. The obtained membership function is denoted as DO in Figure 5.

2. The proposed approach is used to calculated the displacement. This is denoted as P in Figure5.

3. The displacement is approximated considering a first order Taylor expansion of the natural frequencies without considering intervening variables. This is denoted as T in Figure 5. The objective of including this approximation is examining the advantages of including intervening variables for approximating the natural frequencies.

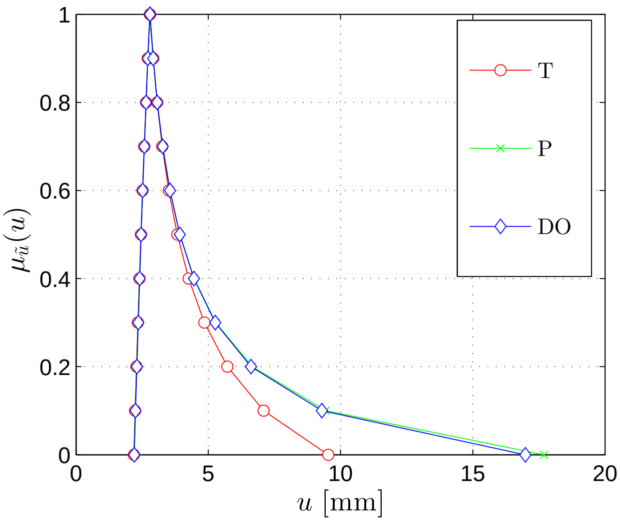

Figure 5. Membership function of amplitude of steady-state vertical displacement.

As noted from Figure 5, the proposed approach is capable of reproducing accurately the sought membership function. In this context, it should be noted that for applying the proposed approach, a single structural analysis plus a sensitivity analysis are required. This is remarkable, as it is possible to approximate the displacement with high quality and reduced numerical costs. In addition, it is seen from the Figure that the approximation T fails in approximating the displacement accurately. This result highlights the benefit of including intervening variables for approximating the natural frequencies.

6. Conclusions

This contribution has presented an approach for performing fuzzy structural analysis of linear dynamical systems. The key issue of the proposed approach consists in introducing a meta-model of the FRF which considers two different approximation levels. The results obtained in this contribution suggest the proposed approach can produce accurate results with a reduced number of structural analyses.
Although the results presented are most promising, further research efforts are required in order to determine the precise range of application of the proposed approach.

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