Multi-Objective Reliability-Based Design Optimization using Subset Simulation Enhanced by Meta-Models

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Abstract: This paper deals with double-looped reliability-based design optimization (RBDO), in which the system reliability is assessed within the inner loop and a designing process is performed in the outer loop. A common approach expressed as single-objective optimization is transformed into a multi-objective case providing results as an approximation of the Pareto front composed of the compromising solutions between cost and reliability. The double-loop formulation of RBDO provides the most accurate approximation of the Pareto front but is computationally demanding even if advanced simulation techniques are used for rare failure events. Nowadays, a *Subset simulation* is a popular method to obtain an estimate of small failure probabilities. Despite the reduction in evaluation time using a Subset simulation when compared to a crude Monte Carlo method, the computational effort is still high with a complex model as a performance function, e.g. a finite element model. The computational model can be replaced by its surrogate in order to reduce the computational costs. This *meta-model* fits the responses evaluated by the original model for the predetermined data, so called a Design of Experiment (DoE). Since the design variables change with every iteration and a meta-model is utilized for a reliability assessment, the meta-model is trained only in the vicinity of the relevant design variable which makes the meta-model computationally faster and more precise. The DoE is updated by selected points from subset simulation samples with respect to two criteria: first, beneficial samples are located in the vicinity of the limit state, which divides the space into a safe region and a failure domain, and second, these samples should also be placed in the sparsest position of the DoE. The described method is illustrated on a classical RBDO benchmark with two objective functions; the first objective is a cost function to be minimized, the second objective is a structural reliability expressed by a reliability index to be maximized. The quality is assessed by comparison to an asymptotic sampling and a Monte Carlo simulation all with responses obtained by an original model and local meta-models.

Keywords: multi-objective optimization, reliability-based design optimization, radial basis functions meta-models, non-dominated sorting genetic algorithm II, asymptotic sampling, subset simulation.

1. Introduction

Structural optimization is a process that seeks the best design under some predefined constraints. A deterministic model is usually unrealistic due to the uncertain inputs such as material properties, a structural topology, loadings etc. The optimal design with deterministic variables often terminates at a boundary between the failure domain and the safe domain and even a small perturbation in inputs can lead to a fatal failure. For that reason, the model uncertainties are introduced; the parameter uncertainties are associated with the input data whereas the structural uncertainties express that the model need not clearly describe the physics of the problem. Optimization under uncertainties looks into two main tasks; the first task called *robust design optimization* deals with the everyday fluctuations in inputs and provides a design with a minimum price that is less sensitive to small perturbations in inputs; the second task known as *reliability-based design optimization* concentrates on worse-case scenarios and offers an economical design with large safety.

In reliability-based design optimization (RBDO), the most challenging task is to evaluate a probability constraint or a probability objective. For such a reliability assessment, it is required to evaluate a structural response several times with different settings of uncertain parameters and design variables, which is the most computer power demanding part of the problem. It is possible to evaluate a probability of failure analytically only for special types of problems, e.g. by Gauss quadrature approaches, Laplace Approximation approaches, etc., which is limiting for RBDO applications. An exact computation is impractical (Aoues and Chateauneuf, 2010). Approximation techniques such as a First-order reliability method (FORM) (Hasofer and Lind, 1974), and simulation techniques such as crude Monte Carlo (MC) (Metropolis and Ulam, 1949) and variance reduction techniques are commonly used. FORM is very often preferred for its speed and relatively few necessary evaluations of the structural response. The drawback is that the obtained reliability assessment is inaccurate for low probabilities of failure and for highly non-linear problems. There are two preferred formulations of FORM in RBDO, that proceed from a key idea that a reliability index is the shortest distance between the most probable failure point (MPFP) and the origin in a standard normal space and therefore it is an optimization task. First, the reliability index approach (RIA) (Enevoldsen and Sørensen, 1994) is a classical reliability formulation, in which the quadratic form of MPFP is minimized such that a limit state function value is less than or equal to a predefined threshold value. Second, performance measure approach (PMA) (Tu, Choi and Park, 1999) is defined as an inverse approach, in which limit state function is minimized such that the quadratic form of MPFP is equal to a predefined threshold value. This formulation is more stable because it is easier to minimize a complicated objective function subject to a simple constraint. Since the expansion of a computing power, the simulation techniques are becoming popular in RBDO, such as Monte Carlo simulation (Papadrakakis, Lagaros and Plevris, 2005), a subset simulation (Valdebenito and Schuüeller, 2010) or an importance sampling (Beaurepaire et al, 2013).

Reliability-based design optimization can be formulated by two linked loops. An optimizer provides a design in the outer loop, for which a probability of failure is evaluated in the inner loop. The double-looped procedure allows a very accurate safety appraisal of each design without any kind of approximation. However, this formulation suffers from large computational demands if a classical Monte Carlo method is used. Fortunately, advanced simulation techniques such as an asymptotic sampling (Bucher, 2009) and a subset simulation (Au and Beck, 2001) can be used for the reliability assessment and the accuracy can be almost maintained with the drastic computational effort reduction.

Although the speed-up is achievable with the advanced simulation techniques, a structural response can be still quite computationally demanding and an approximation of an original structural model has to be implemented. The original model can be replaced by some model of the original model that has a very similar behaviour; however, it is less time consuming to evaluate. Those models of models are called *meta-models* or *surrogate models* and they require just few evaluations of the costly original function. Those evaluations are then used to create the meta-model. Proper locations for an original model evaluation (called *support points*) have to be chosen properly usually by a *Design of Experiments* (DoE) (Montgomery, 1997; Janouchová and Kučerová, 2013) with support points usually uniformly distributed in the whole design space. The accuracy of a meta-model is dependent on several factors such as a number of dimensions, non-linearity of the structural response model, a number of support points, etc.

The initial DoE can be created by as many support points as many times the original model can be evaluated or the initial DoE can be sparse and adaptively improved (*updated*) in interesting locations, subsequently. A type of an update is dependent on a purpose of the meta-model within the meaning of optimization. A meta-model replacing an original model in an objective function is updated differently from the meta-models replacing original models in constraints. For a meta-model utilization in a reliability assessment in RBDO, several updating methods are published in the literature such as Efficient global reliability analysis (Bichon et al, 2008) or Active Kriging using Monte Carlo simulation (Echard, Gayton and Lemaire, 2011). These updating methods are strictly oriented to Kriging meta-models and their error estimate, however, Kriging is not always the best option for the original model substitution (Jin, Chen and Sompson, 2001).

For a reliability assessment in RBDO, the most important region in any model is the boundary between the safe and the failure domain, called a *limit state.* A good meta-model should be the most precise especially around this boundary. If an initial DoE is sparse and it is subsequently adaptively refined, new support points should be placed partly in the vicinity of the limit state, partly in the location with the highest sparsity of DoE. These requirements can be formulated into multi-objective optimization with two objectives to search an optimal position of the new support points in DoE. The first one is the distance to the limit state; the second one is the space-filling property of the DoE, which is optimized by maximizing the minimal interpoint distance (i.e. the Maximin approach). The optimization routine runs repeatedly and obtained Pareto-optimal solutions are added into the existing DoE to get the updated meta-model, see more details in (Pospíšilová, Myšáková and Lepš, 2013). The benefits of this method are that this procedure is independent on a meta-model type, one global meta-model covers the whole design space and it can be treated as the original model, the update is run in parallel, and the DoE is updated with several support points at the same time. The main drawback is that the whole design space is large and the number of support points in DoE has to be enourmous to obtain the similar accuracy with the original model. It has a consequence in needed computational effort; since more samples are used for the interpolating meta-model construction, the more computational effort is necessary. In (Pospíšilová, and Lepš, 2015), we started to use several local meta-models instead of one global meta-model. Small meta-models are trained for every single design of RBDO. These meta-models are built from samples that are in the vicinity of the proposed design by the optimizer and therefore are relevant to that proposal. Evaluations of these tiny meta-models are very fast and can be repeated as many times as necessary by the sampling method with no limitations. The main drawback of the proposed methodology was that the chosen reliability assessment method, an asymptotic sampling, needed larger meta-models because of sequentially scaling random variables over the standard deviation to predict a reliability index. If selected points in DoE for meta-model construction are too few, an asymptotic sampling requires an extrapolation of the meta-model and the resulting reliability assessment is not trustworthy. This paper deals with the incorporation of another reliability assessment method, a subset simulation, into RBDO enhanced by local meta-models introduced in (Pospíšilová, and Lepš, 2015). We expect the possibility of construction of smaller meta-models for the reliability evaluation and following speed-up of finding the approximation of the Pareto front in RBDO.

2. Methodology

2.1. From Single- to Multi-objective Double-looped Reliability-based Design Optimization

Reliability-based design optimization provides a design that is economical as well as reliable in presence of uncertainties. Many formulations of objective functions have been proposed, however, minimization of the total cost of the structure is a frequently used model comprising the initial cost together with the failure risk. This failure risk is defined as the cost of failure multiplied by the probability of failure and because of obstacles in assessing monetary values to all failure consequences (e.g. placing a monetary value on human life) (Frangopol, 1985), an alternative formulation is possible as

$$\min_{\mathbf{d}\in\mathbb{D}} \quad C(\mathbf{d}), \tag{1}$$

$$\text{s. t.} \quad H_i(\mathbf{d}) \leq 0,\ i = 1, \dots, n_e, \tag{2}$$

$$\beta_j(\mathbf{x},\mathbf{d}) \geq \beta_j^{tol},\ j = 1, \dots, n_p. \tag{3}$$

The objective function $C(\cdot)$ is to be minimized with optimal values of design variables arranged in a vector $\mathbf{d}$ that contains deterministic variables or probability distribution parameters (e.g. the mean of random variables). Design variables are chosen from the design space $\mathbb{D}$. All deterministic constraints $H_i(\cdot)$ has to be less than or equal to zero where i is from 1 to n_e and n_e is the total number of constraints. A generalized reliability index $\beta_j(\cdot)$ has to be greater than or equal to a prescribed tolerable threshold β_j^{tol}; generally, β is defined as an inverse cumulative distribution function of the standard normal distribution $\beta = \Phi^{-1}(1 - p_F)$, where p_F is a probability of failure. The influence of uncertain parameters arranged in a vector $\mathbf{x}$ is taken into consideration in these constraints. A number of events n_p is equal to one in the simplest cases.

Multi-objective formulation of the reliability-based design optimization task provides larger room for a decision maker, whereas single-objective formulation forces a researcher to constrain the search space before an optimization procedure is started. The multi-objective formulation addressed here is stated as follows:

$$\min_{\mathbf{d}\in\mathbb{D}} \quad C(\mathbf{d}), \tag{4}$$

$$\max_{\mathbf{d}\in\mathbb{D}} \quad \beta_j(\mathbf{x},\mathbf{d}),\ j = 1, \dots, n_p, \tag{5}$$

$$\text{s. t.} \quad H_i(\mathbf{d}) \leq 0,\ i = 1, \dots, n_e. \tag{6}$$

The reliability constraint in a single-objective formulation is freed to an objective function in a multi-objective formulation. Together with the first objective function $C(\cdot)$, the multi-objective formulation contains two conflicting objectives now and the trade-off between the structural safety and the cost has to be find as the Pareto front from which the most preferable solution is identified afterwards by a decision maker.

A double-loop approach deals with a designing process in an outer loop; an optimizer proposes a combination of design variables $\mathbf{d}^{(k)}$ in each optimization step k and deterministic constraints $H_i(\mathbf{d}^{(k)})$ together with reliability indexes $\beta_j(\mathbf{x}, \mathbf{d}^{(k)})$ are evaluated. A reliability assessment is appraised for each combination of design variables $\mathbf{d}^{(k)}$ in an inner loop to obtain $\beta_j(\mathbf{x}, \mathbf{d}^{(k)})$. The inner loop consists of N_s evaluations of a structural response $g(\mathbf{x}^{(s)}, \mathbf{d}^{(k)})$ with different values of random variables $\mathbf{x}^{(s)}$, with $s = 1, \ldots, N_s$.

2.2. Subset simulation

A subset simulation (Au and Beck, 2001) is a novel methodology that is based on a formulation of the failure event F as an intersection of the intermediate failure events F_m. The rare event problem is then reformulated as series of more frequent events that are easier to solve. The probability of failure is as follows

$$p_F = \text{Prob}[F_1] \cdot \prod_{m=2}^{M} [F_m | F_{m-1}]. \tag{7}$$

The failure probability of the first intermediate domain is evaluated by a classical Monte Carlo method with hundreds of samples N. These samples are sorted in an ascending order and a limit state function $y = g(\cdot)$ is relaxed such that $\text{Prob}[F_1]$ is equal to a predefined value p_0. The limit state function value of the $(p_0 \cdot N)^{\text{th}}$ sample is a relaxed threshold $y_1^\star$ in sorted Monte Carlo samples. The first $(p_0 \cdot N)$ samples are used as seeds for the simulation of samples from conditional probabilities by a Markov chain Monte Carlo (MCMC) with a modified Metropolis algorithm. In each level m, samples obtained by MCMC are sorted and first $(p_0 \cdot N)$ samples serve as seeds in the $(m+1)^{\text{th}}$ step together with a proper relaxation of the limit state function by value $y_m^\star$. The last level M is reached if the failure probability of the M^{th} step with the original limit state is greater than p_0; in other words, if the $y_M^\star$ changes its sign.

2.3. Radial basis functions model

Radial basis functions model (RBF) approximates a complicated true function as a sum of easier basis functions that are symmetrical and centered on a set of support points (Forrester, 2008). For noise-free data, the model is

$$\hat{y}(\mathbf{x}) = \mathbf{w}^T \boldsymbol{\psi} = \sum_{i=1}^{n_c} w_i \psi(||\mathbf{x} - \mathbf{c}^{(i)}||), \tag{8}$$

where $\mathbf{w}$ is a weighting vector, $\boldsymbol{\psi}$ is a vector of length n_c holding evaluated basis functions on Euclidean distances between the prediction $\mathbf{x}$ and centres of basis functions $\mathbf{c}$. Basis functions can be fixed or parametric and are chosen by user according to the type of the original function; frequently used basis functions are Gaussian, Hardy multiquadrics, Inverse multiquadrics, C^n Matérn etc. We use Gaussian parametric bases

$$\psi(r) = \exp\left(\frac{-r^2}{2\sigma^2}\right) \tag{9}$$

where parameters σ are found by an optimization algorithm. Centres of basis functions can be made identical with the Design of Experiments $\mathbf{c}^{(i)} = \mathbf{x}^{(i)}$ and parameters in a weighting vector are evaluated from the system of linear equations $\mathbf{\Psi w} = \mathbf{y}$ where each element of Gram matrix $\mathbf{\Psi}$ is $\Psi_{i,j} = \psi(||x^{(i)} - x^{(j)}||),\ i,j = 1,\ldots,n$; n is a number of points in the DoE, $\mathbf{y}$ is a vector of responses evaluated with the original model or function and $\mathbf{w}$ is again a weighting vector.

2.4. Adaptive retrieval of new support points

A whole methodology is depicted in Figure 1 and it is taken over from (Pospíšilová, and Lepš, 2015). First of all, primary Design of Experiment $\mathcal{Q}$ is constructed. In our previous work (Pospíšilová, Myšáková and Lepš, 2013), we have found out that Halton sequences (Kocis, 1997) are efficient, fast and well distributed, thus we used them for every uniform design in this work. The true response function $\mathrm{g}(\cdot)$ that should be computationally very expensive in real-life optimization problems (e.g. FEM simulation) is used to calculate responses $\mathcal{G}$ for DoE $\mathcal{Q}$. Therefore the DoE should be relatively sparse to save up the computational time. This data-set is saved as Dataset 1 $[\mathcal{Q},\mathcal{G}]$ for the following meta-modeling. The optimizer proposes one individual $\mathbf{d}^{(k)}$ or a set of individuals (in case of population algorithm) in each generation k. In this paper, every individual has a meaning of the mean of a random variable. For each individual, deterministic functions as well as the reliability assessment are evaluated. Deterministic functions are not so computationally problematic for the most cases and if so, they are calculated only once for each individual. The reliability assessment, however, complicates the computational speed. Since we use a sampling approach for probability of failure estimation, we calculate the response function repeatedly. The response function is therefore substituted by a surrogate model, that is uniquely built by $\mathscr{S}(\cdot)$ for each individual $\mathbf{d}^{(k)}$ as $\overline{g}(\cdot)$, thus every individual has its own meta-model and individuals do not share them. The meta-model is set-up from DoE data-set points $\mathcal{Q}$ that are nearby the individual $\mathbf{d}^{(k)}$; the selection of the dataset is labeled as $\mathbf{Q}^{\mathbf{d}^{(k)}}$. We use the *k-nearest neighbors algorithm*[1] (Friedman, 1977) for the proximity assessment where only the prescribed number of points have to be specified. We found out that one tenth of the all DoE points is sufficient but this setting vary from problem to problem. However, the tenth is a good initial guess. The estimation of the probability of failure $p_F(\mathbf{d}^{(k)})$ as well as a value of the cost function $C(\mathbf{d}^{(k)})$ are given back to the optimization algorithm. In case that we do not want to utilize the knowledge about the problem provided by the meta-model and reflect it into the DoE, the optimization algorithm will continue up to the N_k step in the described way.

The reliability simulation method samples around the design variable and the reliability assessment needs the response in these samples. The difference between the response and the limit value of the admissible occurrence is called *limit state function* (LSF) in this paper. The most substantial part of the limit state function is the *limit state*, i.e. the frontier between the safe and failure region,

[1] We use knn-search implemented in Matlab.

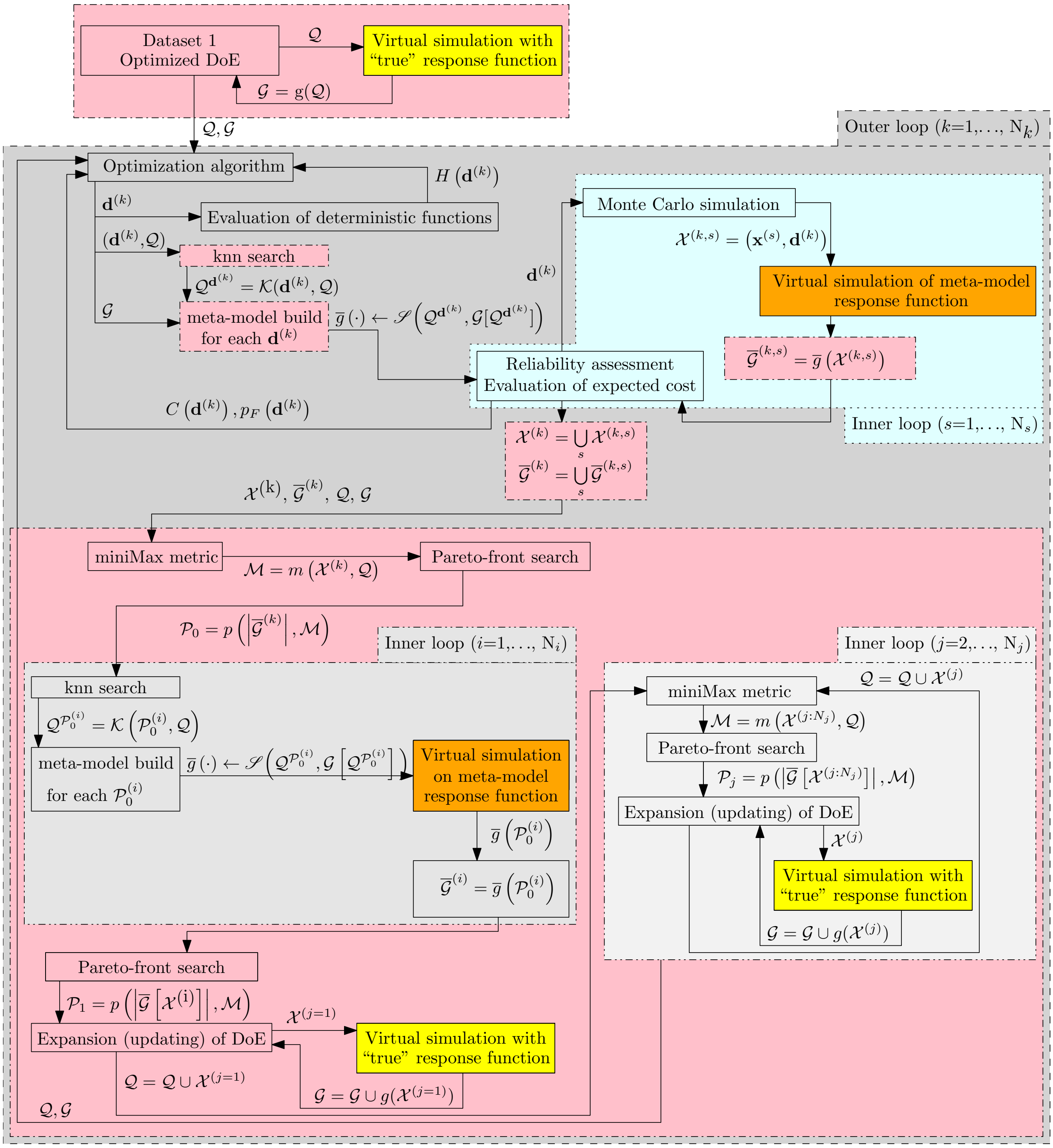

Figure 1. Schematic representation of a double-loop RBDO problem extended to the meta-model update.

sometimes called the *limit state surface*. The meta-model has to be very exact here. Generally, the accuracy of the meta-model is increased by addition of new samples into the DoE $\mathcal{Q}$ including their "true" responses of the original model. Since the original model is still not the solution of the real life problem, we use the quotation marks here. However, this response is more reliable than the meta-model response. New samples to DoE have to be added into the vicinity of the limit state; this is the first criterion. We already know the meta-model responses of the Monte Carlo samples and therefore we choose those new updating points from the Monte Carlo sample set. Every single sample from the reliability sampling method and its response value are therefore stored into $\mathcal{X}^{(k)}$ and $\overline{\mathcal{G}}^{(k)}$, respectively. These new samples should be also placed into the area with the deficient knowledge to uncover more possible limit states. For this second criterion, the miniMax metric m is used, which is the distance to the closest sample in sense of a diameter of the maximum inscribed hypersphere, and it is evaluated and stored in $\mathcal{M}$ for all Monte Carlo samples $\mathcal{X}^{(k)}$ with regard to the original DoE $\mathcal{Q}$. These two criteria are antagonistic, therefore the Pareto front $\mathcal{P}_0$ of the best incomparable solutions is calculated. It is possible to update DoE with the Pareto set (the corresponding coordinates in the design space for the Pareto front), however, the response of the meta-model has not to be accurate enough. Therefore, new meta-model is made for each point in the Pareto set so that the knn-search is carried out again and the one tenth of the closest points is chosen for the meta-model construction. These meta-models are therefore used to get a better meta-model response for the Pareto set. Since the distances to the limit state were recalculated, the Pareto front has to be found again as $\mathcal{P}_1$. In case that the meta-modeling response was good enough, the Pareto front $\mathcal{P}_1$ is the same as previous $\mathcal{P}_0$. In other case, the Pareto set is reduced. The first point of the Pareto set $\mathcal{P}_1$ is added into the final DoE $\mathcal{Q}$ and the "true" model response $g(\cdot)$ is calculated. Since the DoE is changed, the miniMax metric is again recalculated. The second criterion for the update is changed, therefore the new Pareto front $\mathcal{P}_j$ has to be found. If the j^{th} point of the $\mathcal{P}_1$ is still in the Pareto front $\mathcal{P}_j$, it is embedded into the DoE $\mathcal{Q}$ and the "true" response for this sample is computed. The DoE is therefore extended in every k^{th} step of the optimization algorithm.

3. Examples

This example utilizing mathematical functions was chosen to demonstrate the efficiency of the proposed methodology. The study of the single-objective optimization behaviour utilizing a surrogate model was carried out by (Lee and Jung, 2008) and the study of the decoupling approach by (Chen et al, 2013). The optimization problem of the original single-objective optimization problem is to

$$\min \quad f_1 = (\mu_{X_1} - 3.7)^2 + (\mu_{X_2} - 4)^2 \tag{10}$$

$$\text{s.t.} \quad g_1(\mathbf{X}) = -X_1 \sin(4X_1) - 1.1X_2 \sin(2X_2) \tag{11}$$

$$g_2(\mathbf{X}) = X_1 + X_2 - 3 \tag{12}$$

Stochastic variables X_1 and X_2 have a normal distribution with the mean values μ_{X_1} and μ_{X_2}, respectively, and both have a standard deviation equal to 0.1. They are statistically independent.

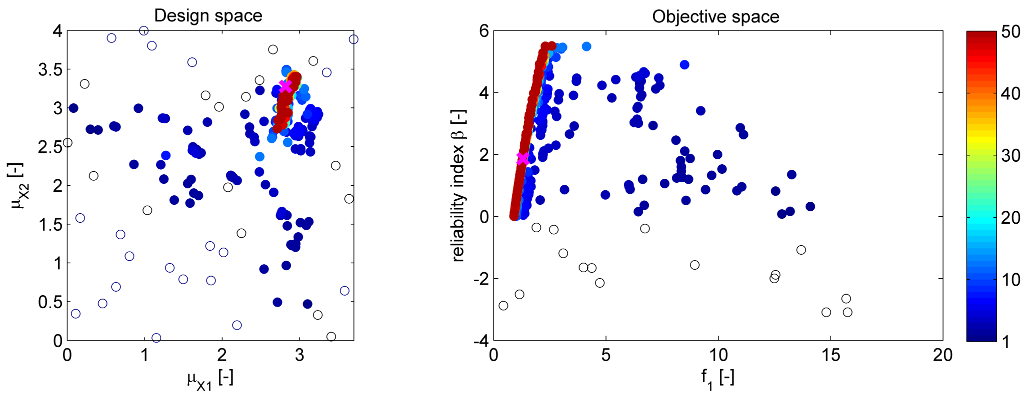

Figure 2. Optimization progression for 50 individuals and 50 generations. Every generation has a different shade depicted in the color bar on the right; the darkest blue color is used for the initial generation, claret wine for the very last generation. The Pareto front was constrained from both sides for β index from 0 to 5.5 that correspond to a failure probability 0.5 and approx. $2 \cdot 10^{-8}$, respectively. Those solutions that did not fulfilled those constraints are depicted with empty circles. The magenta cross is the single-objective optimum found by (Chen et al, 2013), which lies on the Pareto front.

Design variables are represented by both mean values μ_{X_1} and μ_{X_2}. They are in ranges of $0 \leq \mu_{X_1} \leq 3.7$ and $0 \leq \mu_{X_2} \leq 4$, respectively.

This single-objective optimization problem can be reformulated as a multi-objective one considering minimization of the first objective function and maximization of the reliability index of the given mathematical problem

$$\min \quad f_1 = (x_1 - 3.7)^2 + (x_2 - 4)^2 \tag{13}$$

$$\max \quad \beta \tag{14}$$

$$\text{considering LSF} \quad \min \begin{pmatrix} -x_1 \sin(4x_1) - 1.1x_2 \sin(2x_2) \\ x_1 + x_2 - 3 \end{pmatrix} \tag{15}$$

$$0 \leq x_1 \leq 3.7, \quad 0 \leq x_2 \leq 4. \tag{16}$$

The algorithm utilizing local meta-models described in the previous section was used to obtain a final approximation of the Pareto set and the Pareto front. We use Non-dominated sorting genetic algorithm II (Deb, 2002) as a basis multi-objective algorithm with a simulated binary crossover operator (Deb, 1995), a Gaussian mutation operator and a tournament selection operator; probabilities of the operator utilization for creating offspring populations are 0.9, 0.5 and 1, respectively. The number of individuals was set to 50 as well as the number of total generations. The progress of the optimization is depicted in Figure 2. The convergence to the approximated the Pareto-optimal set and front is relatively fast; 20 generations would be sufficient from total 50 generations.

These approximations are compared with resulting the Pareto sets and the Pareto fronts gained by different computational methods in Figure 3. The squares symbols serve for visualisation of meta-model utilization while dots symbols are used for the original analytical function application. The analytical function was affordable for reliability-based design optimization in this benchmark

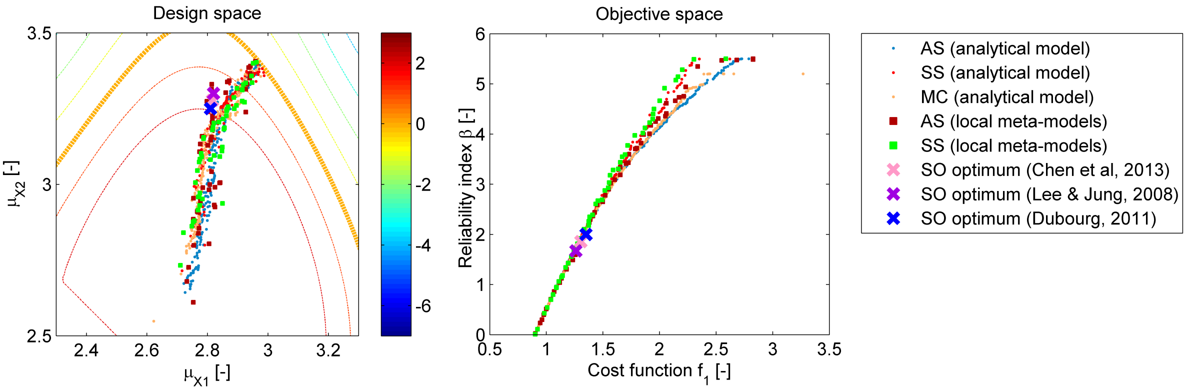

Figure 3. Comparison of Pareto set approximations (left) and Pareto front approximations (right) by several computational methods. The results without use of any meta-models are depicted by dots; an Asymptotic sampling (AS, blue dots), a Subset simulation (SS, red dots), and a Monte Carlo simulation (MC, orange dots) were used to approximate the reliability index. The results with use of local meta-models are drawn by squares; an Asymptotic sampling (claret squares), and a Subset simulation (green squares) were again utilized to approximate the reliability index. Crosses represent single objective optima found in literature; the pink cross is for the optimum taken from (Chen et al, 2013), the violet cross represents the optimum from (Lee and Jung, 2008), and the blue cross depicts the optimum taken from (Dubourg, 2011).

because its evaluation is fast; in case that a structural response is evaluated via a finite element model, a meta-model utilization is inevitable for reasonable results. The orange dot symbols represent an approximation that should be the most precise since a Monte Carlo simulation together with the original analytical function was used. A setting for the number of samples for a Monte Carlo method was contingent on the failure probability pre-evaluation $p_{F,AS}$ by an Asymptotic sampling. This probability $p_{F,AS}$ was then used to predict a number of samples for a Monte Carlo simulation $100/p_{F,AS}$ with a coefficient of variation approximately 10%. Since we used Halton sequences instead of an ordinary pseudo-random number generator, the predicted coefficient of variation should be even smaller. The visible distinction between individual methods is particularly in the upper part of the Pareto front for higher numbers of a reliability index; up to β-index equal to 3, there is no dramatic difference between individual reliability methods as well as between the analytical model and the meta-model utilization. The cross symbols serve for visualisation of single-objective optima found in literature; the original problem was restricted to the reliability index equal to two.

If the analytical model is compared with meta-models utilization and the same reliability assessment method is used, an asymptotic sampling variant has the largest discrepancy in the upper part of the Pareto front. The principal problem is in an interconnection between the asymptotic sampling and local meta-models. Large reliability indices require larger number of scaling steps of standard deviations and therefore a meta-model has to be more complex or it has to extrapolate. A problematic situation is depicted in Figure 4 on the left. Almost the whole design space is included in the sampling space for a β-index prediction and since only a small neighbourhood of the mean values was selected to construct a local meta-model, the resulting reliability index is unreliable. Therefore, a subset simulation was employed together with local meta-models since it makes use of

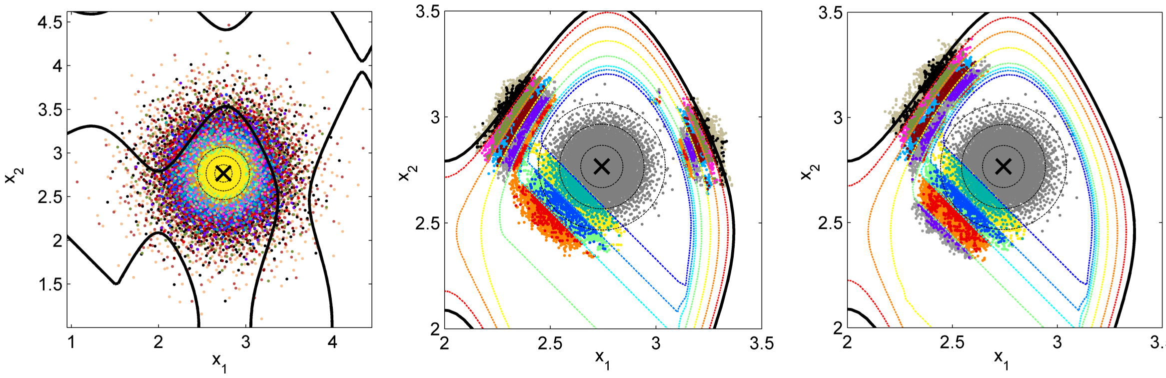

Figure 4. Comparison of sampling methods for a reliability assessment in the same mean values of variables, $\mu_{X1} = 2.74$ and $\mu_{X2} = 2.71$ (the black cross). The bold solid line is the limit state, black dashed circles mark the mean values plus one, two, and three standard deviations, respectively. An asymptotic sampling (left) needed standard deviation scaling 13 times, samples with the same standard deviation are depicted with the same colour. A subset simulation (middle, right) shows instability in selection of sampling directions. Colours of samples serve for differentiation of levels as well as with their seeds. Coloured dashed contours distinguish different $y^{\star}_{m}$ threshold levels. Each subset simulation assessment comprised eight levels.

narrower neighbourhood for meta-model training in which the only problem is the instability of the selection of the subsequent random walks directions, see Figure 4 in the middle and on the right. A subset simulation method proceeds in the steepest descent direction. A fixed portion of samples is taken from samples generated in the previous step and sorted in the ascending order serves for initial seeds in a random walk. Since a pseudo-random generator sampler does not always hit all possible directions to all most probable failure points, some important region can be omitted. This is again illustrated in Figure 4 middle and right, where two independent runs of a subset simulation with the same setting are shown.

The comparison of adaptively added support points into DoE between subset simulation samples and asymptotic sampling samples is depicted in Figure 5 by grey triangles. A subset simulation (Figure 5 right) uses narrower surroundings around the mean values for the reliability index prediction than an asymptotic sampling. An optimization procedure is moving mostly inside the safe space (depicted by orange and red contours) with the individuals representing mean values of random variables in latter generations; this is the reason for so many points added into the safe domain and around the limit state utilizing a subset simulation samples. On the other hand, an asymptotic sampling utilizes a wider space for the β-index prediction than a subset simulation (Figure 5). It is problematic for local meta-models but profitable for exploration of possible limit state locations. An update utilizing a subset simulation samples omitted the part of the limit state on the right of the design space where a potential optima can be placed. Nevertheless, the exploration by a subset simulation samples was sufficient for this benchmark and resembling Pareto sets were obtained with both simulation methods. The only but crucial difference is in β-indices approximations; a subset simulation using local meta-models seems to be more stable than an asymptotic sampling as depicted in Figure 3.

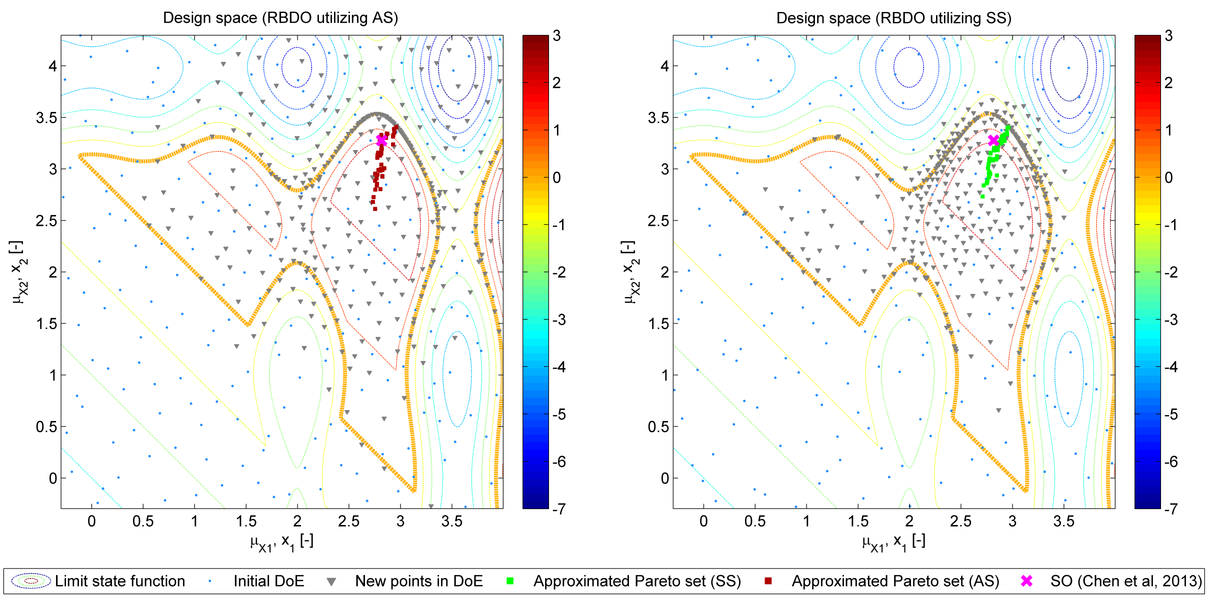

Figure 5. The design space after optimization. An initial Design of Experiments (DoE) is depicted by blue dots whereas adaptively added points to DoE are represented by grey triangles. The red squares (left) or green squares (right) serve for the Pareto set approximation selected from the last generation of NSGA-II; a single objective optimum taken from (Chen et al, 2013) used for a comparison is depicted by the magenta cross. The limit state function serves only for illustrative purposes, it was evaluated only in points of DoE. Two reliability assessment methods are compared; an Asymptotic sampling (left) enables to explore the design space in breath whereas a Subset simulation (right) concentrates samples added to DoE more narrowly around the individuals from an optimization algorithm.

4. Conclusions

Reliability-based design optimization searches for a design that is both economical and reliable. Since reliability index approximation requires plenty of structural responses computations, the most challenging task in RBDO is to evaluate a probability constraint or a probability objective. Even if any advanced simulation method is used to predict a reliability index, a meta-model utilization is necessary for designing real-life structures. One global meta-model for the whole design space needs an enormous number of support points to get a response similar to the original model. If more samples are used for the interpolating meta-model construction, the more computational effort is necessary and, moreover, numerical problems can be expected. A candidate design optimum usually does not need a whole design domain to predict safety of this design, and therefore local meta-models can be incorporated to speed-up an optimization of the designing process. We utilized two reliability assessment sampling methods together with local meta-models; an asymptotic sampling in our previous work (Pospíšilová, and Lepš, 2015) and a subset simulation in this paper. Both of these methods approximate a reliability index and a probability of failure and both methods are dependent on the shape of the limit state in the vicinity of the most probable failure point. In an asymptotic sampling, the higher the predicted reliability index is the wider neighbourhood of mean values is employed and thus the larger local meta-model is necessary. For that reason, a subset simulation requiring narrower neighbourhood for a β-index prediction is more profitable together with local meta-models. The resulting Pareto fronts obtained with the original response model and with local meta-models almost overlap themselves, which implies that the meta-models provide sufficient accuracy of response values with beneficial speed-up of a response evaluation.

Another speed-up can be obtained via parallel evaluations. Since the proposed method is multi-level, partitioning is also available in different levels. Although we use a Non-dominated sorting genetic algorithm II, the algorithm is still population-based and reliability indices can be evaluated for each individual in parallel. Several different strategies can be used such as global parallelization, coarse grains, and fine grains, which differ in communication inside a population (Alba and Tomassini, 2002). The partitioning in population level is inherent but load balancing is complicated because of different computational efforts for different values of reliability indices. Since we use sampling methods for a reliability assessment, these samples can also be evaluated in parallel. A subset simulation utilizes a modified Metropolis algorithm with plenty of short Markov chains. All these chains can be evaluated again in parallel. Since each level in a subset simulation is dependent on the previous level exclusive of the first level, the partitioning via levels is not straightforward. An asymptotic sampling uses several Monte Carlo simulations with diverse standard deviations of random variables in Gaussian space. All these Monte Carlo levels contain a small number of samples. The minimum number of the levels is set up in advance, therefore the partitioning is possible in whole Monte Carlo simulations. The number of samples in every Monte Carlo level is known in advance as well and therefore, every Monte Carlo sample can be evaluated in parallel. The partitioning inside a sampling reliability assessment method is beneficial since the load to each threat can be decomposed with equal modular unit.

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